Problem 1:

$$\begin{cases} u_{tt} = \Delta u & 0 \le x \le a, \quad 0 \le y \le b \\ u_x(t,0,y) = u_x(t,a,y) = 0 \\ u(t,x,0) = u(t,x,b) = 0 \\ u(0,x,y) = f(x,y), \quad u_t(0,x,y) = 0 \end{cases}$$

Separation of variables yields

$$\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2$$

$$\begin{cases} \dfrac{X''}{X} = -\mu^2 \, ; \quad X'(0) = X'(a) = 0 \end{cases} \longrightarrow X_n(x) = \cos\left(\frac{\pi n}{a} x\right)$$

$$\mu_n = \frac{n\pi}{a}, \; n = 0, 1, \ldots$$

$$\to \begin{cases} \dfrac{Y''}{Y} = \mu_n^2 - \lambda^2 \\ Y(0) = Y(b) = 0 \end{cases} \longrightarrow \mu_n^2 - \lambda^2 = -\left(\frac{m\pi}{b}\right)^2, \; m = 1, 2, \ldots$$

$$Y_n(y) = \sin\left(\frac{m\pi}{b} y\right)$$

$$\to \lambda_{n,m} = \sqrt{\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2} \quad \text{and} \quad T''_{n,m} + \lambda^2_{n,m} T = 0$$

$$\longrightarrow T_{n,m}(t) = A_{n,m}\cos(\lambda_{n,m} t) + B_{n,m}\sin(\lambda_{n,m} t)$$

$$\to u(t,x,y) = \sum_{n=0}^{+\infty} \sum_{m=1}^{+\infty} \left(A_{n,m}\cos(\lambda_{n,m} t) + B_{n,m}\sin(\lambda_{n,m} t)\right) \cdot \cos\left(\frac{\pi n}{a} x\right) \sin\left(\frac{m\pi}{b} y\right)$$

Now

$$f(x,y)=\sum_{n=0}^{+\infty}\sum_{m=1}^{+\infty} A_{n,m}\cos\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)$$

$$0=g(x,y)=\sum_{n=0}^{+\infty}\sum_{m=1}^{+\infty} B_{n,m}\cdot\lambda_{n,m}\cos\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)$$

Use the formulas on p. 50 to find

$$A_{0,m}=\frac{2}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{m\pi}{b}y\right)dx\,dy$$

$$A_{n,m}=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{m\pi}{b}y\right)\cos\left(\frac{n\pi}{a}x\right)dx\,dy$$

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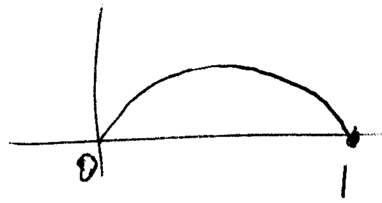
$$B_{0,m}=\frac{2}{\lambda_{n,m}ab}\int_0^a\int_0^b g(x,y)\sin\left(\frac{m\pi}{b}y\right)dx\,dy=0$$

$$B_{n,m}=\frac{4}{\lambda_{n,m}ab}\int_0^a\int_0^b g(x,y)\sin\left(\frac{m\pi}{b}y\right)\cos\left(\frac{n\pi}{a}x\right)dx\,dy=0$$

## Problem 2:

First compute $f^*(x), g^*(x)$. We have that

$f(x) = \sin(\pi x) \longrightarrow$ 

It follows that
$$f^*(x) = \begin{cases} |\sin(\pi x)| & \text{for } x > 0 \\ -|\sin(\pi x)| & x < 0 \end{cases}$$

Similarly,
$$g^*(x) = \begin{cases} -5 & x > 0 \\ 5 & x < 0 \end{cases}$$

$$\rightarrow \quad u(x,t) = \frac{1}{2}\left[f^*(x-t) + f^*(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g^*(s)\,ds$$

<u>Case 1:</u> $x > t \rightarrow$
$$u(x,t) = \frac{1}{2}\left(|\sin\pi(x-t)| + |\sin\pi(x+t)|\right) + \frac{1}{2}(-5)\left(\cancel{x}+t - (\cancel{x}-t)\right)$$

<u>Case 2:</u> $x < t \rightarrow$
$$u(x,t) = \frac{1}{2}\left[-|\sin(\pi(x-t))| + |\sin\pi(x+t)|\right] + \frac{1}{2}\int_{t-x}^{t+x}(-5)\,dx$$
$$= \frac{1}{2}\left[|\sin\pi(x+t)| - |\sin\pi(x-t)|\right] - \frac{5}{2}\left((\cancel{t}+x) - (\cancel{t}-x)\right)$$

Problem 3:

We set up the eigenvalue problem

$$\begin{cases} u_{xx} = \lambda u & 0<x<1 \\ u(0)=u(1)=0 \end{cases}$$

$$\leadsto \quad \lambda_n = -\pi^2 n^2, \quad u_n(x) = \sin(n\pi x)$$

Therefore, in order to solve the PDE, we seek its solution in the form

$$u(x,y,t) = \sum_{n=1}^{+\infty} a_n(y) \sin(n\pi x)$$

We have
$$\Delta u = \sum_{n=1}^{+\infty} (a_n'' - n^2\pi^2 a_n) \sin(n\pi x)$$

$$1 = \sum_{n=1}^{+\infty} b_n \sin(n\pi x), \quad b_n = 2\int_0^1 \sin(n\pi x)\,dx$$

$$\boxed{b_n = \frac{2}{n\pi}\left((-1)^n - 1\right)}$$

Thus, $\Delta u = 3u - 1$ is equivalent to

$$\sum_{n=1}^{+\infty} \left(a_n''(y) - n^2\pi^2 a_n(y)\right) \sin(n\pi x) = \sum_{n=1}^{+\infty} \left(3a_n(y) - b_n\right) \sin(n\pi x)$$

$$\longrightarrow \quad a_n''(y) - n^2\pi^2 a_n(y) = 3a_n(y) - b_n \qquad n = 1, 2, \ldots$$

That is: $\begin{cases} a_n'' - (n^2\pi^2+3)a_n = -b_n \\ \end{cases}$

What about boundary conditions?

$$1 = u(x,0) = \sum_{n=1}^{+\infty} a_n(0)\sin(n\pi x) \longrightarrow \begin{matrix} a_n(0) = b_n \\ a_n'(\cancel{0}) = b_n \end{matrix}$$

$$1 = u_y(x,\cancel{0}) = \sum_{n=1}^{+\infty} a_n'(\cancel{0})\sin(n\pi x)$$

(Recall $b_n$ is the Fourier coeff. in the sin expansion)

Thus, we are solving the second order ODE with inhomogeneous term $\underline{\underline{(-b_n)}}$.

Particular solution is $a_n^0 = \dfrac{b_n}{n^2\pi^2+3}$

$$\longrightarrow a_n(y) = A_1 \cdot e^{(\sqrt{n^2\pi^2+3})y} + A_2 e^{-(\sqrt{n^2\pi^2+3})y} + \frac{b_n}{n^2\pi^2+3}$$

? $A_1, A_2$ ? $\begin{cases} A_1 + A_2 + \dfrac{b_n}{n^2\pi^2+3} = b_n \\ \cancel{\sqrt{n^2\pi^2+3}(A_1 - A_2) + \dfrac{b_n}{n^2\pi^2+3} = b_n} \\ \sqrt{n^2\pi^2+3}\left[A_1 \cdot e^{\sqrt{n^2\pi^2+3}} - A_2 \cdot e^{-\sqrt{n^2\pi^2+3}}\right] + \dfrac{b_n}{n^2\pi^2+3} = b_n \end{cases}$

$\longrightarrow$ solve for $\boxed{A_1, A_2}$!

Problem 4:

$$\begin{cases} \Delta u = 0 & 0 \le x \le a, \quad 0 \le y \le b \\ u_x(0,y) = u_x(a,y) = 0 \\ u_y(x,b) = 0 \\ u_y(x,0) + u(x,0) = f(x) \end{cases}$$

Note: typo in the exam.

Separation of variables:

$$u = X(x) \cdot Y(y)$$

$$\begin{cases} \dfrac{X''}{X} = -\dfrac{Y''}{Y} = -\lambda^2 \\ X'(0) = X'(a) = 0 \\ Y'(b) = 0 \end{cases} \longrightarrow \lambda_n = \frac{n\pi}{a}, \; n = 0, 1, \ldots \qquad X_n = \cos\left(\frac{n\pi}{a} x\right)$$

Now

$$\begin{cases} Y_n'' - \lambda_n^2 Y = 0 \\ Y_n'(b) = 0 \end{cases} \longrightarrow Y_n(y) = \cosh(\lambda_n (y-b))$$

$$\longrightarrow u(x,y) = \sum_{n=0}^{+\infty} A_n \cos\left(\frac{n\pi}{a} x\right) \cosh\left(\frac{n\pi}{a}(y-b)\right)$$

$$f(x) = u_y(x,0) + u(x,0) = \sum_{n=0}^{+\infty} \left( A_n \cosh\left(\frac{n\pi b}{a}\right) + \frac{n\pi}{a} A_n \sinh\left(\frac{n\pi b}{a}\right)\right) \cos\left(\frac{n\pi}{a} x\right)$$

$$\longrightarrow \quad A_0 = \frac{1}{a} \int_0^a f(x)\,dx \, ;$$

$$A_n \left( \cosh\left(\frac{n\pi b}{a}\right) - \frac{n\pi}{a} \sinh\left(\frac{n\pi b}{a}\right)\right) = \frac{2}{a} \int_0^a f(x) \cos\left(\frac{n\pi x}{a}\right) dx$$

$\longrightarrow$ Solve for $A_n$ (Note resonances!!!)