MATH 810: PROJECT V
SOLUTIONS

(1) Exercise 13.1/p. 104
Hint: Consider $\nu^{\pm}$, so that $\nu = \nu^+ - \nu^-$ and show that for the measures $\nu^{\pm}$, one has $\nu^+ << \mu$ and $\nu^- << \mu$.
Solution:
Consider the Hahn decomposition of ν $X = E \cup F$, where E and F are the negative and the positive sets respectively. Take a measurable set A, so that $\mu(A) = 0$. It follows that $\mu(A \cap E) = \mu(A \cap F) = 0$.
We have by $\nu << \mu$, $\nu(A \cap E) = \nu(A \cap F) = 0$. Thus,

$$\nu^+(A) = \nu(A \cap E) = 0, \nu^-(A) = \nu(A \cap F) = 0.$$

Thus $\nu^{\pm} << \mu$. By Radon-Nykodim,

$$\nu^{\pm}(A) = \int_A f^{\pm} d\mu.$$

Finally,

$$\nu(A) = \nu^+(A) - \nu^-(A) = \int (f^+ - f^-) d\mu = \int_A f d\mu.$$

(2) Prove the following extension of Exercise 15.19. For a function g, measurable with respect to $\mathcal{A} \times \mathcal{B}$, prove that

$$\| \|g\|_{L^r(d\nu_y)} \|_{L^q(d\mu_x)} \leq \| \|g\|_{L^q(d\mu_x)} \|_{L^r(d\nu_x)}$$

or equivalently

$$\left(\int \left(\int |g(x,y)|^r d\nu(y) \right)^{q/r} d\mu(x) \right)^{1/q} \leq \left(\int \left(\int |g(x,y)|^q d\mu(x) \right)^{r/q} d\mu(x) \right)^{1/r}$$

whenever $0 < r \leq q \leq \infty$.
Hint: Note that Exercise 15.19 corresponds to the case $r = 1$. Here, reduce to Exercise 15.19, with $p = \frac{q}{r} \geq 1$.
Solution:
When $q < \infty$, we have by Exercise 15.19

$$\left(\int \left(\int |g(x,y)|^r d\nu(y) \right)^{q/r} d\mu(x) \right)^{r/q} \leq \int \left(\int |g(x,y)|^q d\mu(x) \right)^{r/q} d\nu(y).$$

Raising both sides by $1/r$, we obtain the desired inequality.
If $q = \infty, r < \infty$, we have for every x_0

$$|g(x_0, y)|^r \leq \sup_x |g(x,y)|^r$$

Integrating this in y gives us

$$\left(\int |g(x_0, y)|^r dy\right)^{1/r} \leq \left(\int \sup_x |g(x,y)|^r dy\right)^{1/r}$$

Taking essential supremum in x_0 yields

$$\| \|g\|_{L^r_y} \|_{L^\infty_x} \leq \| \|g\|_{L^\infty_x} \|_{L^r_y},$$

as stated. For $q = \infty, r = \infty$, we have equality, as both norms are equal to $\|g\|_{L^\infty_{xy}}$.

(3) Prove the generalized Hölder's inequality. That is

$$\int |f_1(x)| \dots |f_n(x)| d\mu \leq \|f_1\|_{L^{p_1}} \dots \|f_n\|_{L^{p_n}},$$

whenever $1 \leq p_1, p_2, \dots, p_n \leq \infty : \frac{1}{p_1} + \dots \frac{1}{p_n} = 1$.
Hint: The case $n = 2$ is the standard Hölder's inequality. Then, argue by induction. Assuming that it is true for any $p_1, \dots, p_{n-1} : \frac{1}{p_1} + \dots \frac{1}{p_{n-1}} = 1$, consider $1 \leq p_1, p_2, \dots, p_n \leq \infty : \frac{1}{p_1} + \dots \frac{1}{p_n} = 1$. First, deal with the easy case $p_n = \infty$. For $p_n < \infty$, define $q_n : \frac{1}{q_n} + \frac{1}{p_n} = 1$. By the standfard Hölder's

$$\int |f_1(x)| \dots |f_n(x)| d\mu \leq \left(\int (|f_1(x)| \dots |f_{n-1}(x)|)^{q_n} d\mu\right)^{1/q_n} \|f_n\|_{L^{p_n}}.$$

Note that $\frac{1}{p_1} + \dots \frac{1}{p_{n-1}} = \frac{1}{q_n}$ and continue the chain of inequalities above.
Solution:
For $p_n = \infty$, we have

$$\int |f_1(x)| \dots |f_n(x)| d\mu \leq \sup_{|} f_n(x)| \int |f_1(x)| \dots |f_{n-1}(x)| d\mu$$

to which we apply the Hölder's inequality for $n-1$ functions. We obtain

$$\int |f_1(x)| \dots |f_n(x)| d\mu \leq \|f_n\|_{L^\infty} \|f_1\|_{L^{p_1}} \dots \|f_{n-1}\|_{L^{p_{n-1}}}$$

where $1 = \frac{1}{p_1} + \dots \frac{1}{p_{n-1}}$.

If $p_n < \infty$, we have

$$\int |f_1(x)| \dots |f_n(x)| d\mu \leq \left(\int (|f_1(x)| \dots |f_{n-1}(x)|)^{q_n} d\mu\right)^{1/q_n} \|f_n\|_{L^{p_n}}.$$

We apply the Hölder's with $1 = \frac{1}{p_1/q_n} + \dots + \frac{1}{p_{n-1}/q_n}$. We have

$$\int (|f_1(x)| \dots |f_{n-1}(x)|)^{q_n} d\mu \leq (\int |f_1|^{p_1})^{q_n/p_1} \dots (\int |f_{n-1}|^{p_{n-1}})^{q_n/p_{n-1}}$$

Thus raising to power $1/q_n$,

$$(\int (|f_1(x)| \dots |f_{n-1}(x)|)^{q_n} d\mu)^{1/q_n} \leq \|f_1\|_{L^{p_1}} \dots \|f_{n-1}\|_{L^{p_{n-1}}}$$

Putting everything together yields the estimate.

(4) Recall that $f * g(x) = \int_{R^1} f(y)g(x-y)dy$. Prove the Young's inequality

$$\|f * g\|_{L^q} \leq \|f\|_{L^p}\|g\|_{L^r},$$

whenever $1 \leq p, q, r < \infty$ and $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$.

Hint: Check that the conditions ensure that $\frac{p}{q} + \frac{p}{r'} = 1 = \frac{r}{q} + \frac{r}{p'}$ and hence, one can write

$$|f * g(x)| \leq \int |f(y)|^{p/r'}[|f(y)|^{p/q}|g(x-y)|^{r/q}]|g(x-y)|^{r/p'}dy.$$

Note also $\frac{1}{r'} + \frac{1}{q} + \frac{1}{p'} = 1$.

Solution:

Apply Hölder's inequality to the last estimate. We have

$$\begin{aligned} |f * g(x)| &\leq \int |f(y)|^{p/r'}[|f(y)|^{p/q}|g(x-y)|^{r/q}]|g(x-y)|^{r/p'}dy \leq \\ &\leq (\int |f(y)|^p dy)^{1/r'}(\int |f(y)|^p|g(x-y)|^r dy)^{1/q}(\int |g(x-y)|^r dy)^{1/p'} = \\ &= \|f\|_{L^p}^{p/r'}\|g\|_{L^r}^{r/p'}(\int |f(y)|^p|g(x-y)|^r dy)^{1/q}. \end{aligned}$$

Raising to power q and integrating in x yields

$$\int |f * g(x)|^q \leq \|f\|_{L^p}^{qp/r'}\|g\|_{L^r}^{qr/p'} \int\int |f(y)|^p|g(x-y)|^r dydx.$$

By Fubini's

$$\begin{aligned} \int\int |f(y)|^p|g(x-y)|^r dydx &= \int |f(y)|^p(\int |g(x-y)|^r dx)dy = \|g\|_{L^r}^r \int |f(y)|^p dy = \\ &= \|g\|_{L^r}^r\|f\|_{L^p}^p. \end{aligned}$$

Thus,

$$\|f * g\|_{L^q}^q \leq \|f\|_{L^p}^{qp/r'}\|g\|_{L^r}^{qr/p'}\|g\|_{L^r}^r\|f\|_{L^p}^p = \|f\|_{L^p}^{p+qp/r'}\|g\|_{L^q}^{r+qr/p'}.$$

But according to the relations between p, q, r, $1/r' = 1/p - 1/q$, $1/p' = 1/r - 1/q$. So,

$$\begin{aligned} p + qp/r' &= p(1 + q/r') = p(1 + q(1/p - 1/q)) = q \\ r + qr/p' &= r(1 + q/p') = r(1 + q(1/r - 1/q)) = q, \end{aligned}$$

whence

$$\|f * g\|_{L^q}^q \leq \|f\|_{L^p}^q\|g\|_{L^q}^q,$$

as stated.