MATH 810: PROJECT V
DUE: DEC. 9TH, 2014

(1) Exercise 13.1/p. 104
Hint: Consider $\nu^{\pm}$, so that $\nu = \nu^+ - \nu^-$ and show that for the measures $\nu^{\pm}$, one has $\nu^+ << \mu$ and $\nu^- << \mu$.

(2) Prove the following extension of Exercise 15.19. For a function g, measurable with respect to $\mathcal{A} \times \mathcal{B}$, prove that

$$\|\|g\|_{L^r(d\nu_y)}\|_{L^q(d\mu_x)} \leq \|\|g\|_{L^q(d\mu_x)}\|_{L^r(d\nu_x)}$$

or equivalently

$$\left(\int \left(\int |g(x,y)|^r d\nu(y)\right)^{q/r} d\mu(x)\right)^{1/q} \leq \left(\int \left(\int |g(x,y)|^q d\mu(x)\right)^{r/q} d\mu(x)\right)^{1/r}$$

whenever $0 < r \leq q \leq \infty$.
Hint: Note that Exercise 15.19 corresponds to the case $r = 1$. Here, reduce to Exercise 15.19, with $p = \frac{q}{r} \geq 1$.

(3) Prove the generalized Hölder's inequality. That is

$$\int |f_1(x)| \dots |f_n(x)| d\mu \leq \|f_1\|_{L^{p_1}} \dots \|f_n\|_{L^{p_n}},$$

whenever $1 \leq p_1, p_2, \dots, p_n \leq \infty : \frac{1}{p_1} + \dots \frac{1}{p_n} = 1$.
Hint: The case $n = 2$ is the standard Hölder's inequality. Then, argue by induction. Assuming that it is true for any $p_1, \dots, p_{n-1} : \frac{1}{p_1} + \dots \frac{1}{p_{n-1}} = 1$, consider $1 \leq p_1, p_2, \dots, p_n \leq \infty : \frac{1}{p_1} + \dots \frac{1}{p_n} = 1$. First, deal with the easy case $p_n = \infty$. For $p_n < \infty$, define $q_n : \frac{1}{q_n} + \frac{1}{p_n} = 1$. By the standfard Hölder's

$$\int |f_1(x)| \dots |f_n(x)| d\mu \leq \left(\int (|f_1(x)| \dots |f_{n-1}(x)|)^{q_n} d\mu\right)^{1/q_n} \|f_n\|_{L^{p_n}}.$$

Note that $\frac{1}{p_1} + \dots \frac{1}{p_{n-1}} = \frac{1}{q_n}$ and continue the chain of inequalities above.

(4) Recall that $f * g(x) = \int_{R^1} f(y) g(x-y) dy$. Prove the Young's inequality

$$\|f * g\|_{L^q} \leq \|f\|_{L^p} \|g\|_{L^r},$$

whenever $1 \leq p, q, r < \infty$ and $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$.
Hint: Check that the conditions ensure that $\frac{p}{q} + \frac{p}{r'} = 1 = \frac{r}{q} + \frac{r}{p'}$ and hence, one can write

$$|f * g(x)| \leq \int |f(y)|^{p/r'} [|f(y)|^{p/q} |g(x-y)|^{r/q}] |g(x-y)|^{r/p'} dy.$$

Note also $\frac{1}{r'} + \frac{1}{q} + \frac{1}{p'} = 1$.