MATH 810: PROJECT IV
DUE: DEC. 5TH 2017

(1) 12.8/page 98
(2) 13.8/page 105
(3) Prove that if $p, q : \frac{1}{p} + \frac{1}{q} \neq 1$, then, the inequality

$$|\int_A f(x)g(x)dx| \leq C\|f\|_{L^p(A)}\|g\|_{L^q(A)}, \tag{1}$$

cannot hold for any $C > 0$, if $A = \mathbf{R}^1$ or $A = (0, \infty)$. On the other hand, prove (1) for $A = (0, 1)$ and $p, q : \frac{1}{p} + \frac{1}{q} \geq 1$.
Hint: Scaling - apply the inequality to $f_\lambda(x) := f(\lambda x)$.
(4) 15.7/page 142.
(5) In 15.7/page 142: Find $p_0 \in (1, 2)$, so that $f \in L^p : p > p_0$ we have $\int_0^1 f(x)g_n(x)dx \to 0$, while for $p < p_0$, we have an example $f \in L^p$, so that $\int_0^1 f(x)g_n(x)dx$ does not converge to zero.
(6) 15.10/page 142
Hint: First show that it is enough to show the claim for a dense set of g, for example g is a bounded function in L^q, with compact support. Next, use that $f_n \to f$ a.e., means that $f_n \to f$ in measure on the support of g (which is compact set, with finite measure).
(7) 15.15/page 143. Here, we do need finite measure space $\mu(X) < \infty$.
Hint: It is easy to see that if the sum is finite, then $f \in L^p(\mu)$. For the reverse direction, write

$$\begin{aligned}\int |f(x)|^p d\mu &= \int_{|f(x)|<1} |f(x)|^p d\mu + \sum_{n=0}^{\infty} \int_{|f(x)|\in[2^n,2^{n+1})} |f(x)|^p d\mu \sim \\ &\sim \int_{|f(x)|<1} |f(x)|^p d\mu + \sum_{n=0}^{\infty} 2^{np}\mu(\{|f(x)| \in [2^n, 2^{n+1}\}).\end{aligned}$$

On the other hand,

$$\sum_{n=1}^{\infty} 2^{np}\mu(\{|f(x)| > 2^n\}) = \sum_{n=1}^{\infty} 2^{np} \sum_{k=0}^{\infty} \mu(\{|f(x)| \in [2^{n+k}, 2^{n+k+1})\}).$$

So, try to show the equivalence of the two sums.
(8) 15.23/page 144. Quantify the p (i.e. estimate the p in terms of α).
Hint: Consider $A = \{x : |f(x)| \in (2^n, 2^{n+1})\}$.
(9) 15.25/page 145 (This is a simple version of the Schur's test).
Hint: For the L^p bounds, you may use the duality $(L^p)^* = L^q$:

$$\|Tf\|_{L^p} = \sup_{\|g\|_{L^q}=1} \int K(x, y)f(y)g(x)dxdy.$$

There are also other ways to do it.