MATH 810: PROJECT IV
DUE: NOV. 13TH, 2014

(1) Problem 11.4/page 88;
Solution:
We have by Fubini's

$$\begin{aligned} 1 = m(A) &= \int_{[0,1]^2} \chi_A(x,y)dxdy = \int_0^1 \left(\int_0^1 \chi_A(x,y)dy\right) dx = \\ &= \int_0^1 m_1(s_x(A))dx \leq \max_{x\in[0,1]} m_1(s_x(A)) \leq 1. \end{aligned}$$

It follows that there all equalities above. Thus, $m_1(s_x(A)) = 1$ a.e. Indeed, assume not. Then, there exists $\delta > 0$, so that

$$m(B_\delta) = m(x \in (0,1) : \{m_1(s_x(A)) < 1 - \delta\}) > 0.$$

Then, by Chebyshev's

$$\begin{aligned} 1 &= \int_0^1 m_1(s_x(A))dx = \int_{B_\delta} m_1(s_x(A))dx + \int_{B_\delta^c} m_1(s_x(A))dx \leq \\ &\leq m(B_\delta)(1-\delta) + m(B_\delta^c) < m(B_\delta) + m(B_\delta^c) = 1. \end{aligned}$$

Thus, $1 < 1$, a contradiction.

(2) Problem 11.11/page 89
Hint: Use Fubini's

$$\int(\int f(x,y)d\mu y)dx = \int(\int f(x,y)dx)d\mu(y),$$

for specifically designed function f.
—bf Solution:
Consider the function $f(x,y) = \chi_{(0,c]}(y-x)$ and its integral over the product space $(\mathbf{R}^1, dx) \times (\mathbf{R}^1, dm(y))$, where dx is the Lebesgue measure on $\mathbf{R}^1$. We have by Fubini,

$$\begin{aligned} &\int\int \chi_{(0,c]}(y-x)dxd\mu(y) = \int(\int \chi_{(0,c]}(y-x)d\mu(y))dx = \\ &= \int(\int \chi_{(x,x+c]}(z)d\mu(z))dx = \int \mu(x,x+c])dx. \end{aligned}$$

On the either hand, again by Fubini,

$$\begin{aligned} &\int\int \chi_{(0,c]}(y-x)dxd\mu(y) = \int(\int \chi_{(0,c]}(y-x)dx)d\mu(y) = \\ &= \int(\int_{y-c}^y dx)d\mu(y) = c\int_{\mathbf{R}^1} d\mu(y) = c\mu(\mathbf{R}^1). \end{aligned}$$

This is the statement, if one realize that $f(x+c) - f(x) = \mu(x, x+c])$.

(3) Problem 12.4/page 98 (The definition of absolute continuity is actually on page 99).
Solution:
The definition for absolute continuity for signed measure is nowhere to be found in the book, but it is the usual thing - we say that the signed measure ν is abs. cont. w.r.t. the (complete) measure μ (denoted $\nu << \mu$), if $\mu(A) = 0$ implies $\nu(A) = 0$. Suppose that.
Let $\nu = \nu^+ - \nu^-$ be the Hahn decomposition of ν. Let E, F be the negative and positive sets respectively for ν. Then, by definition $\nu^-(A) = \nu(A \cap E)$, $\nu^+(A) = \nu(A \cap F)$. We have that $A \cap F \subset A$ and $A \cap E \subset F$. Since μ is complete $\mu(A \cap F) = 0, \mu(A \cap E) = 0$ and hence by the assumption $\nu << \mu$, $\nu(A \cap F) = 0 = \nu(A \cap E)$. Thus,
$$\nu^+(A) = \nu(A \cap F) = 0, \nu^-(A) = \nu(A \cap E) = 0.$$
Conversely, if $\nu^\pm << \mu$, we have that if $\mu(A) = 0$, $\nu^\pm(A) = 0$ and hence
$$\nu(A) = \nu^+(A) - \nu^-(A) = 0.$$

(4) Problem 12.7/page 98
Solution:
We only prove it for μ^+, the other statement is similar. We have
$$\mu^+(A) = \mu(A \cap F) \leq \sup\{\mu(B) : B \in \mathcal{A}, B \subset A\}.$$
Conversely, take any $B \in \mathcal{A}, B \subset A$. We have
$$\mu(B) = \mu^+(B) - \mu^-(B) \leq \mu^+(B) \leq \mu^+(A),$$
where we have used that $\mu^-(B) \geq 0$ and then, since μ^+ is a measure, $\mu^+(B) \leq \mu^+(A)$, since $B \subset A$. Thus,
$$\sup\{\mu(B) : B \in \mathcal{A}, B \subset A\} \leq \mu^+(A).$$

(5) Problem 12.8/page 98
Solution:
Again, we show this, by establishing a two way inequality.
$$|\mu|(A)| = \mu^+(A) + \mu^-(A) = \mu(A \cap F) - \mu(A \cap E) =$$
$$= \quad |\mu(A \cap F)| + |\mu(A \cap E)| \leq \sup\{\sum |\mu(B_j)| : B_j \subset \mathcal{A}, B_j \cap B_k = \emptyset, \cup B_j = A\}$$
Conversely, for $B_j \subset \mathcal{A}, B_j \cap B_k = \emptyset, \cup B_j = A$, we have
$$|\mu(B_j)| = |\mu^+(B_j) - \mu^-(B_j)| \leq |\mu^+(B_j)| + |\mu^-(B_j)| = |\mu|(B_j).$$
Hence
$$\sum |\mu(B_j)| \leq \sum |\mu|(B_j) = |\mu|(A),$$
since $|\mu|$ is a measure. Thus,
$$\sup\{\sum |\mu(B_j)| : B_j \subset \mathcal{A}, B_j \cap B_k = \emptyset, \cup B_j = A\} \leq |\mu|(A).$$