MATH 810: PROJECT III
DUE: OCT. 28TH, 2014

(1) Let E be a Lebesgue measurable subset of $\mathbf{R}^1$, $m(E) < \infty$. Prove that

$$\lim_{x\to 0} m(E \cap E + x) = m(E).$$

Here $E + x = \{x + y : y \in E\}$.
Hint: Use the definition of $m(E) = \inf\{\sum m(Q_j) : E \subset \cup Q_j\}$. In addition, you can take the infimum to be over finite unions only.
Solution:
Fix $\epsilon > 0$. Then, find $O = \cup_{j=1}^n Q_j$, so that Q_j are disjoint intervals, $E \subset O$

$$m(E) + \epsilon > m(O),$$

whence $m(O \setminus E) < \epsilon$. Next, $O \cap O + x \setminus E \cap E + x \subset O \setminus E \cup \{O + x\} \setminus \{E + x\}$. Thus,

$$\begin{aligned} |m(E \cap E + x) - m(O \cap O + x)| &\leq m(O \setminus E) + m(\{O + x\} \setminus \{E + x\}) = \\ &= 2m(O \setminus E) < 2\epsilon. \end{aligned}$$

Finally, for x small enough $Q_j + x$ is disjoint from $Q_i : i \neq j$. Thus,

$$O + x \cap O = \cup_{j=1}^n (Q_j \cap Q_j + x)$$

and because of the finite sum,

$$\lim_{x\to 0} m(O + x \cap O) = \lim_{x\to 0} \sum_{j=1}^n m(Q_j \cap Q_j + x) = \sum_{j=1}^n m(Q_j) = m(O).$$

So, there exists $\delta = \delta(\epsilon)$, so that for $|x| < \delta$,

$$m(O + x \cap O) > m(O) - \epsilon.$$

Thus, for $|x| < \delta$,

$$m(E) \geq m(E \cap E + x) \geq m(O \cap O + x) - 2\epsilon \geq m(O) - 3\epsilon > m(E) - 3\epsilon.$$

This shows

$$\lim_{x\to 0} m(E \cap E + x) = m(E).$$

(2) Let f_n be a sequence of nonnegative measurable extended (i.e. could take values of $+\infty$) real-valued functions defined on a measure space $(X, \mathcal{A}, \mu)$. Suppose there is an integrable function g on x, so that $f_n(x) \leq g(x)$. Prove that

$$\int_X \limsup f_n d\mu \geq \limsup \int_X f_n d\mu.$$

Hint: Fatou's lemma.
Solution:

Consider the functions $g_n := g(x) - f_n(x) \geq 0$. Apply the Fatou's lemma to them. We get

$$\liminf \int g_n(x) d\mu \geq \int \liminf g_n(x) d\mu$$

But

$$\begin{aligned} \liminf \int g_n(x) d\mu &= \int g(x) d\mu - \limsup \int f_n(x) d\mu \\ \int \liminf g_n(x) d\mu &= \int g(x) d\mu - \int \limsup f_n(x) d\mu \end{aligned}$$

Resolving the last inequalities yields the result.

(3) Exercise 10.2
Hint: For $d(f_n, f) \to 0 \Rightarrow f_n \to f$ in measure, use the Chebyshev's inequality. For the reverse direction, split the integration in the definition of $d(f_n, f)$ over the set $A_{n,\epsilon} = \{x : |f_n(x) - f(x)| > \epsilon\}$ and its complement.
Solution:
The fact that d is a metric was shown in class. Suppose that $d(f_n, f) \to 0$. Let $\epsilon > 0$. Since the function $t \to \frac{t}{1+t}$ is increasing, we have by Chebyhev's

$$\frac{\epsilon}{1+\epsilon} m(\{x : |f_n(x) - f(x)| > \epsilon\}) \leq d(f_n, f)$$

Thus, by the squeeze theorem

$$\lim_{n \to \infty} m(\{x : |f_n(x) - f(x)| > \epsilon\}) = 0.$$

Suppose now that $\lim_n m(A_n^\epsilon) = 0$. We have

$$\begin{aligned} d(f_n, f) &= \int \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} d\mu = \int_{A_n^\epsilon} \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} d\mu + \\ &+ \int_{(A_n^\epsilon)^c} \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} d\mu \leq \mu(A_n^\epsilon) + \epsilon \mu((A_n^\epsilon)^c) \leq \mu(A_n^\epsilon) + \epsilon \mu(X) \end{aligned}$$

Choosing $\delta > 0$, we first select $\epsilon : \epsilon\mu(X) < \delta/2$. Then, we select $N = N(\epsilon) = N(\delta)$, so that for $n > N$, we have $\mu(A_n^\epsilon) < \delta/2$. Thus, for $N > N(\delta)$, we have

$$d(f_n, f) < \delta.$$

(4) Let f_n be a sequence of positive Lebesgue measurable functions on $[0, 1]$, so that

$$\sum_{n=1}^{\infty} m(\{x \in [0, 1] : f_n(x) > 1\}) < \infty.$$

Show that $\limsup_n f_n(x) \leq 1$ a.e.
Hint: Consider the set $A = \{x : \limsup f_n(x) > 1\}$. Show that

$$A \subset \cap_{k=1}^{\infty} \cup_{n=k}^{\infty} \{x \in [0, 1] : f_n(x) > 1\})$$

and argue from there.
Solution:
We use the formula $f(x) = \limsup f_n(x) = \inf_{k\geq 1} \sup_{n\geq k} f_n(x)$. Hence,

$$A \subset \cap_{k=1}^{\infty}\{x : \sup_{n\geq k} f_n(x) > 1\} = \cap_{k=1}^{\infty} \cup_{n=k}^{\infty} \{x : f_n(x) > 1\}.$$

Thus,

$$m(A) = \lim_k m(\cup_{n=k}^{\infty}\{x : f_n(x) > 1\}) \leq \lim_k \sum_{n=k}^{\infty} m(\{x : f_n(x) > 1\}).$$

The Cauchy condition for the convergence of the series $\sum_{n=1}^{\infty} m(\{x \in [0,1] : f_n(x) > 1\})$ dictates that

$$\lim_k \sum_{n=k}^{\infty} m(\{x : f_n(x) > 1\}) = 0.$$

Thus, $m(A) = 0$.

(5) Suppose that $g : [0,1] \to \mathbf{R}^1$ is a bounded and measurable function. Suppose that $\int_0^1 f(x)g(x)dm = 0$ for all continuous functions $f : \int_0^1 f(x)dm = 0 = \int_0^1 xf(x)dm$. Prove that there exists C_1, C_2, so that $g(x) = C_1 + C_2 x$ a.e.
Hint:
- Prove first the "easy" version. That is, assuming $\int_0^1 f(x)g(x)dm = 0$ for all continuous functions f implies that $g(x) = 0$ a.e.
- For the function g, introduce C_1, C_2, to be the unique solution of

$$\left| \begin{array}{l} C_1 + \frac{C_2}{2} = \int_0^1 g(x)dm \\ \frac{C_1}{2} + \frac{C_2}{3} = \int_0^1 xg(x)dm \end{array} \right.$$

Prove that the function $\tilde{g}(x) := g(x) - C_1 - C_2 x$ satisfies $\int_0^1 h(x)\tilde{g}(x)dm = 0$ for all continuous functions h. Conclude.
How do we come up with this system for C_1, C_2?
Solution:
The easy version was proved in class. Note that the function $\tilde{g}$ satisfies $\int \tilde{g}(x)dm = 0 = \int x\tilde{g}(x)dm$.
Next, take any continuous function h and consider the two constants C_1^h, C_2^h, defined as solution to

$$\left| \begin{array}{l} C_1^h + \frac{C_2^h}{2} = \int_0^1 h(x)dm \\ \frac{C_1^h}{2} + \frac{C_2^h}{3} = \int_0^1 xh(x)dm \end{array} \right.$$

Note that the function $\tilde{h} := h(x) - C_1^h - C_2^h x$ satisfies $\int \tilde{h}(x)dm = \int \tilde{h}(x)xdm = 0$ as well. We have

$$\int_0^1 h(x)\tilde{g}(x)dm = \int_0^1 \tilde{h}(x)\tilde{g}(x)dm - C_1^h \int \tilde{g}(x)dm - C_2^h \int x\tilde{g}(x)dm =$$

$$= \int_0^1 \tilde{h}(x)\tilde{g}(x)dm = \int_0^1 \tilde{h}(x)g(x)dm - C_1^g \int_0^1 \tilde{h}(x)dm - C_2^g \int_0^1 x\tilde{h}(x)dm =$$

$$= \int_0^1 \tilde{h}(x)g(x)dm = 0,$$

where in the last identity, we have used the assumption on g. Thus, by the simple version, we have that $\tilde{g}(x) = 0$ a.e. Thus, $g(x) = C_1^g + C_2^g x$ a.e.