# MATH 810: PROJECT III
# DUE: OCT. 28TH, 2014

(1) Let $E$ be a Lebesgue measurable subset of $\mathbf{R}^1$, $m(E) < \infty$. Prove that
$$\lim_{x \to 0} m(E \cap E + x) = m(E).$$
Here $E + x = \{x + y : y \in E\}$.
**Hint:** Use the definition of $m(E) = \inf\{\sum m(Q_j) : E \subset \cup Q_j\}$. In addition, you can take the infimum to be over finite unions only.

(2) Let $f_n$ be a sequence of nonnegative measurable extended (i.e. could take values of $+\infty$) real-valued functions defined on a measure space $(X, \mathcal{A}, \mu)$. Suppose there is an integrable function $g$ on $x$, so that $f_n(x) \leq g(x)$. Prove that
$$\int_X \limsup f_n d\mu \geq \limsup \int_X f_n d\mu.$$
**Hint:** Fatou's lemma.

(3) Exercise 10.2
**Hint:** For $d(f_n, f) \to 0 \Rightarrow f_n \to f$ in measure, use the Chebyshev's inequality. For the reverse direction, split the integration in the definition of $d(f_n, f)$ over the set $A_{n,\epsilon} = \{x : |f_n(x) - f(x)| > \epsilon\}$ and its complement.

(4) Let $f_n$ be a sequence of positive Lebesgue measurable functions on $[0, 1]$, so that
$$\sum_{n=1}^{\infty} m(\{x \in [0, 1] : f_n(x) > 1\}) < \infty.$$
Show that $\limsup_n f_n(x) \leq 1$ a.e.
**Hint:** Consider the set $A = \{x : \limsup f_n(x) > 1\}$. Show that
$$A \subset \cap_{k=1}^{\infty} \cup_{k=n}^{\infty} \{x \in [0, 1] : f_n(x) > 1\})$$
and argue from there.

(5) Suppose that $g : [0, 1] \to \mathbf{R}^1$ is a bounded and measurable function. Suppose that $\int_0^1 f(x)g(x)dm = 0$ for all continuous functions $f : \int_0^1 f(x)dm = 0 = \int_0^1 xf(x)dm$. Prove that there exists $C_1, C_2$, so that $g(x) = C_1 + C_2 x$ a.e.
**Hint:**
- Prove first the "easy" version. That is, assuming $\int_0^1 f(x)g(x)dm = 0$ for all continuous functions $f$ implies that $g(x) = 0$ a.e.
- For the function $g$, introduce $C_1, C_2$, to be the unique solution of
$$\left| \begin{array}{l} C_1 + \frac{C_2}{2} = \int_0^1 g(x)dm \\ \frac{C_1}{2} + \frac{C_2}{3} = \int_0^1 xg(x)dm \end{array} \right.$$

Prove that the function $\tilde{g}(x) := g(x) - C_1 - C_2 x$ satisfies $\int_0^1 h(x)\tilde{g}(x)dm = 0$ for all continuous functions $h$. Conclude.
How do we come up with this system for $C_1, C_2$?