MATH 810: PROJECT II
DUE: OCT. 9TH, 2014

(1) Exercise 5.1/page 44
Solution:
We need to check that $\{x : f(x) > a\}$ is measurable for each real a. Let $r_n \to a, r_n > a$. We have that
$$\{x : f(x) > a\} = \cup_{n=1}^{\infty} \{x : f(x) > r_n\}$$
and hence $\{x : f(x) > a\}$ is measurable.

(2) Exercise 5.2/page 44
Hint: Cover $(0, 1)$ with a countable family of intervals, so that on each one of them $f = g_j$.
Solution:
For each x, take the $r_x > 0$ and a Borel g_x, so that $f = g_x$ on $(x - r_x, x + r_x)$. Thus $(0, 1) = \cup_x (x - r_x, x + r_x)$. We can use the Lindelöf property of $(0, 1)$, which states that there is a countable subcover[1]. That is, there is $\{x_j\}$, so that
$$(0, 1) \subset \cup_{j=1}^{\infty} (x_j - r_j, x_j + r_j)$$
with corresponding functions g_j. Now for any real a, we have that
$$f^{-1}(a, \infty) = \cup_{j=1}^{\infty} \big((x_j - r_j, x_j + r_j) \cap g_j^{-1}(a, \infty)\big)$$
Since g_j are Borel, it follows that $f^{-1}(a, \infty)$ is Borel as well.

(3) Let $(X, \mathcal{A})$ be a measure space. Let $f : X \to [-\infty, \infty]$ (that is f may be unbounded for some $x \in X$). Prove that f is measurable if and only if $f^{-1}(\{-\infty\}) \in \mathcal{A}, f^{-1}(\{\infty\}) \in \mathcal{A}$ and $f : Y \to \mathbf{R}^1$ is measurable, where $Y = f^{-1}(-\infty, \infty)$.
Solution:
By the definition of Borel functions, $f : X \to [-\infty, \infty]$ is Borel, if and only $f^{-1}(A)$ is Borel set in X, whenever A is a Borel subset of $[-\infty, \infty]$. But A is a Borel subset of $[-\infty, \infty]$ if and only if $A = B$ or $A = B \cup \{-\infty\}$ or $A = B \cup \{\infty\})$ or $A = B \cup \{\infty\}) \cup \{-\infty\}$, where B is Borel in $(-\infty, \infty)$. Clearly now, $f^{-1}(A)$ is Borel if and only if $f^{-1}(\{-\infty\})$, $f^{-1}(\{\infty\})$ and $f^{-1}(B)$ are all Borel.

(4) Exercise 6.4/page 50. You may use the linearity of the integral
Hint: Let $\epsilon > 0$. Consider a simple function s, so that $0 \le s(x) \le |f(x)|$
$$\int |f(x)| d\mu < \int s(x) d\mu + \frac{\epsilon}{2}.$$

[1]This actually can be easily proved - just used the compactness of $[1/n, 1 - 1/n]$ and cover by finitely many intervals and then $(0, 1) = \cup_n [1/n, 1 - 1/n]$

Solution: The simple function s defined above is bounded, say $M = sup_X s(x)$. Denote $\delta := \frac{\epsilon}{2M}$. Let $A : \mu(A) < \delta$. We have by $s(x) \leq |f(x)|$ that

$$\int_{A^c} s(x) d\mu \leq \int_{A^c} |f(x)| d\mu$$

Thus

$$\begin{aligned} \int_A f(x) d\mu &= \int |f(x)| d\mu - \int_{A^c} |f(x)| d\mu \leq \int s(x) d\mu + \frac{\epsilon}{2} - \int_{A^c} s(x) d\mu = \\ &= \int_A s(x) d\mu + \frac{\epsilon}{2} \leq M\mu(A) + \frac{\epsilon}{2} < \epsilon, \end{aligned}$$

if we take into account $\mu(A) < \delta = \frac{\epsilon}{2M}$.

(5) Let $(X, \mathcal{A}, \mu)$ is measure space $f \geq 0$ and measurable. Show that

$$\lambda(A) := \int_A f d\mu,$$

defined for $A \in \mathcal{A}$, is a measure. Also, show that for each $g \geq 0$, measurable, we have

$$\int g d\lambda = \int f g d\mu.$$

Hint: For the measure part, use Proposition 7.5. For the integral fromula, prove it first for simple functions. Extending to arbitrary positive functions, it is easy to show $\int g d\lambda \leq \int f g d\mu$. For the reverse inequality, use the approximation result Proposition 5.14 and the monotone convergence theorem.

Solution:

λ is a measure, because for each disjoint family of sets $\{A_j\} : A = \cup_j A_j$, we have by Proposition 7.5

$$\lambda(A) = \int_A f d\mu = \sum_j \int_{A_j} f = \sum_j \lambda(A_j)$$

Next, for $g = \sum_i a_i \chi_{A_i}$ a simple function, we have

$$\int g d\lambda = \sum_i a_i \lambda(A_i) = \sum_i a_i \int_{A_i} f d\mu = \int (\sum_i \chi_{A_i}) f d\mu = \int f g d\mu.$$

By the definition of the integral and the previous step,

$$\int g d\lambda = \sup\{\int s d\lambda : 0 \leq s \leq g\} = \sup\{\int s f d\mu : 0 \leq s \leq g\} \leq \int f g d\mu,$$

where in the last step, we have used that $sf \leq fg \Rightarrow \int s f d\mu \leq \int f g d\mu$.

In the reverse direction, approximate $s_n \uparrow g$. It follows that $s_n f \uparrow g f$ and hence by Lebesgue dominated convergence $\int s_n f d\mu \to \int g f d\mu$. But then, again by $\int g d\lambda = \sup\{\int s d\lambda : 0 \leq s \leq g\}$

$$\int g d\lambda \geq \limsup_n \int s_n d\lambda = \limsup_n \int s_n f d\mu = \int g f d\mu.$$

It follows that $\int g d\lambda = \int f g d\mu$.