MATH 810: PROJECT I
DUE: SEPT. 16, 2014

(1) Exercise 2.7, page 11.
Solution: Clearly $\emptyset, X \in \mathcal{A}$, because $0 = \chi_\emptyset, 1 = \chi_X$ are in the set $\mathcal{F}$. Next, we need to verify that if $A_j \in \mathcal{A}$, then $\cup_{j=1}^\infty A_j \in \mathcal{A}$. We have that $\chi_{A_j} \in \mathcal{F}$. Define $B_1 = A_1, B_j = A_j \setminus (B_1 \cup \ldots \cup B_{j-1})$.
We prove by induction that $B_j \in \mathcal{A}$. Indeed, by induction hypothesis $K_j := B_1 \cup \ldots \cup B_{j-1} \in \mathcal{A}$, i.e. $\chi_{K_j} \in \mathcal{F}$. So

$$\chi_{B_j} = \chi_{A_j}(1 - \chi_{K_j}) \in \mathcal{F}.$$

Now, since $\cup_{j=1}^\infty A_j = \cup_{j=1}^\infty B_j$ and the B_j are disjoint, we have

$$\chi_{\cup_{j=1}^\infty A_j}(x) = \sum_{j=1}^\infty \chi_{B_j}(x) \in \mathcal{F}.$$

as a limit of the partial sums $s_N(x) = \sum_{j=1}^N \chi_{B_j}(x) \in \mathcal{F}$.

(2) Exercise 3.8, page 16:
Hint: Follow the steps outlined below.
(a) Show that $\sigma(\mathcal{A} \cup \mathcal{N}) = \{A \cup N : A \in \mathcal{A}, N \in \mathcal{N}\}$. This is nontrivial!
(b) Define $\tilde{\mu} : \mathcal{B} \to [0, \infty]$,

$$\tilde{\mu}(A \cup N) := \mu(A).$$

(c) Prove that $\tilde{\mu}$ is a measure (i.e. σ additivity).
(d) Prove that all null sets for $\tilde{\mu}$ are in $\sigma(\mathcal{A} \cup \mathcal{N})$, i.e. $(X, \sigma(\mathcal{A} \cup \mathcal{N}), \tilde{\mu})$ is a complete measure space.

Solution: Since $\sigma(\mathcal{A} \cup \mathcal{N})$ is a σ algebra, which contains $\mathcal{A} \cup \mathcal{N}$, it remains to prove that $\mathcal{A} \cup \mathcal{N}$ is a σ algebra itself and then, it would be equal to $\sigma(\mathcal{A} \cup \mathcal{N})$. Clearly, $X, \emptyset$ are in $\mathcal{A} \cup \mathcal{N}$. Next, complements are preserved as well. Indeed, let B be a set of measure zero, so that $N \subset B$. We have

$$(A \cup N)^c = A^c \cap N^c = A^c \cap (B^c \cup B \setminus N).$$

But $A^c \cap B^c \in \mathcal{A}$, since $\mathcal{A}$ is a σ algebra. Also, $A^c \cap B \setminus N$ is a null set as a subset of B. Finally, we need to check for σ additivity. We have

$$\cup_{j=1}^\infty (A_j \cup N_j) = (\cup_{j=1}^\infty A_j) \cup (\cup_{j=1}^\infty N_j).$$

The first set belongs to $\mathcal{A}$ since it is σ algebra, while for the second take $B_j : N_j \subset B_j$ of measure zero. Then $\cup_{j=1}^\infty B_j$ is also a measure zero set and hence $\cup_{j=1}^\infty N_j \in \mathcal{N}$.
For the σ additivity of the measure $\tilde{\mu}$, we have

$$\tilde{\mu}(\cup_{j=1}^\infty (A_j \cup N_j)) = \mu(\cup_{j=1}^\infty A_j) = \sum_j \mu(A_j) = \sum_j \tilde{\mu}(A_j \cup N_j).$$

Finally, Z is a null set for $\tilde{\mu}$, if $Z = A \cup N$, so that $\mu(A) = 0$. But then $A \cup N$ is a null set and hence an element of $\mathcal{N}$, hence it belongs to the σ algebra $\sigma(\mathcal{A} \cup \mathcal{N})$.

(3) Exercise 4.3/page 34

Solution: The property $\mu^*(\emptyset) = 0$ is obvious, just pick $B = \emptyset$. Also, if $A_a \subset A_2$, we have that $\mu^*(A_1) \leq \mu^*(A_2)$, just by arguing that infimum over large set is no bigger than the infimum over small set. Finally, let A_i are arbitrary subsets of X. Let $\varepsilon > 0$ and consider sets $B_i \in \mathcal{A}$, so that $A_i \subset B_i$, but $\mu^*(A_i) + \varepsilon 2^{-i} > \mu(B_i)$. Then, $A := \cup_i A_i \subset \cup_i B_i$. Recalling that for each measure, we have $\mu(\cup_i B_i) \leq \sum_{i=1}^{\infty} \mu(B_i)$, we have

$$\mu^*(A) \leq \mu(\cup_i B_i) \leq \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} (\mu^*(A_i) + 2^{-i}\varepsilon) = \varepsilon + \sum_{i=1}^{\infty} \mu^*(A_i).$$

Since $\varepsilon > 0$ is arbitrary, we have proved

$$\mu^*(A) \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

(4) Exercise 4.10/page 35

Hint: For every $\delta > 0$, there is a family of intervals $\{I_j\}_j$, so that $A \subset \cup I_j$ and

$$m(A) + \delta > \sum_j m(I_j)$$

On the other hand,

$$m(A) \leq \sum_j m(A \cap I_j)$$

Solution: by the definition of a the outer measure, there exists a family of intervals $\{I_j\}_j$, so that $A \subset \cup I_j$ and

$$m(A) + \delta = \mu^*(A) + \delta > \sum_j m(I_j)$$

By the assumption

$$(1 - \varepsilon) \sum_j m(I_j) \geq \sum_j m(A \cap I_j) \geq m(A).$$

Putting everything together, it follows that

$$(1 - \varepsilon)(m(A) + \delta) \geq m(A),$$

which implies $\frac{\delta(1-\varepsilon)}{\varepsilon} \geq m(A)$, which is supposed to be true for all $\delta > 0$ (ε is fixed!). Thus, $m(A) = 0$, in contradiction with the assumptions.