Global regularity for Yang-Mills fields in $\mathbb{R}^{1+5}$

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Received (28 August 2009)  
Revised (8 March 2010)

Communicated by Nader Masmoudi

Abstract. We show global persistence of solutions with small data for the model equation $\Box u = u \cdot \nabla u + u^3$, on $\mathbb{R}^{1+d}$, $d \geq 5$, subject to the Coulomb gauge condition $\sum_{j=1}^d \partial_j u^j = 0$. In particular, this covers the important case of the Yang-Mills problem.

Keywords: Yang-Mills model, singular bilinear operators, parallel interactions

1. Introduction

In this paper, we will be interested in the analytical properties of the Yang-Mills problem in dimensions $d \geq 5$. In order to motivate our discussion, we first examine the geometrical origins of the system. A connection 1-form on $\mathbb{R}^{1+d}$ taking values in the Lie algebra of a compact Lie group $G$, is called Yang-Mills connection, if it is a critical point for the functional

$\langle F, F \rangle = \int \int |F_{j,k}|^2,$

where the curvature tensor is given by $F_{j,k} = \partial_j A_k - \partial_k A_j + [A_j, A_k]$. The Euler-Lagrange equations for the minimizer are then expressed in the form $\sum_{j=0}^d D^j F_{j,k} = 0$, $k = 1, \ldots, d$, where $D_j H = \partial_j H + [A_j, H]$ and $A = (A_0, A_j, j = 1, \ldots, d)$. If we fix the Coulomb gauge condition, we arrive at

$\Box u^j = \sum_{j=1}^d [u^j, \partial_j u^i] + F_i(u) \quad \sum_{j=1}^d \partial_j u^j = 0$  \hspace{1cm} (1.1)

where $F(u)$ is a cubic term.

The abelian case, which is slightly more tractable, corresponds to the Maxwell-Klein-Gordon equation or (MKG) in short. In the Coulomb gauge, one can write
(MKG) as

\[
\Delta A_0 = -\text{Im}(\psi \bar{D}_0 \psi), \\
\Box A_j = -\text{Im}(\psi \bar{D}_j \psi), \quad j = 1, \ldots, d, \\
\Box A \psi = 2iA_0 \partial_t \psi + i\partial_t A_0 \psi - A_0^2 \psi, \\
d\psi(A) = 0.
\]

where \(\Box = \partial_t - \Delta_A\) and \(\Delta_A = \sum_j (\partial_j + iA_j(t, x))^2\) is the magnetic Laplacian. This is a well-known reduction, and we refer the reader to [7], [4], [12], [15], [16], [17] for recent analytical studies and especially the earlier paper [22] for the geometrical justification for the existence of the Coulomb gauges. We record that the energy for the system is given by

\[
\int |\nabla A_0|^2 + |\partial_t A|^2 + |\nabla A|^2 + |\nabla A \psi|^2 \, dx = \text{const.}
\]

and the scale invariant Sobolev space for the corresponding Cauchy problem is \(H^{d/2-1} \times H^{d/2-2}\). Note that in dimension \(d = 4\), we have that both problems are both scaling and energy critical, which makes them especially attractive and interesting.

Before we go on with the main theorem, let us take this opportunity to review the current results about (1.1), (1.2) and some related simplified models. This will be a very incomplete list of results, focusing on the more recent developments. The local well-posedness theory for (1.2), is well settled by now, at least in the higher dimensional case. In the higher dimensional case \(d \geq 4\), local well-posedness was established in [9], for data in the almost critical space \(H^{d/2-1} \times H^{d/2-2+}\). For \(d = 3\), local well-posedness was shown in [7] for \(s \geq 1\), then improved to \(s \geq 3/4\) in [4], and in \(s > 3/4\) [20] in the temporal gauge\(^a\) and recently the almost critical local well-posedness result \(s > 1/2\) was established in [11]. In all of these results, with the exception of [11], the considerations were made for the simpler model problems \(\Box u = |\nabla|^{-1} Q(u, u)\) and \(\Box \phi = Q(|\nabla|^{-1} \phi, \phi)\), where \(Q\) is a linear combination of \(Q_{\epsilon \delta}(u, v) = \partial_i u \partial_j v - \partial_j u \partial_i v\). Ironically, it turns out that the the \(H^s(\mathbb{R}^3)\) local well-posedness result for the model problems associated to (MKG) fails for \(s < 3/4\), while of course \(H^{1/2+}\) l.w.p holds for the full (MKG) problem, [11].

In contrast with the local theory, the global well-posedness theory and in particular the global regularity problem is less well-understood in the intermediate dimensions \(d = 3, 4, 5\). Regarding the (MKG) in high dimensions, we should mention here that the case \(d \geq 6\) was solved in [12] for the (MKG) and recently (again in \(n \geq 6\)) for the full Yang-Mills system, [10]. By assuming additional angular regularity, Sterbenz has been able to show global regularity results for the more general equations \(\Box u = u \partial_t u\) in dimensions \(d \geq 4\), [15], [16], [17]. Here, the smallness is assumed in the following norm\(^b\)

\[
\| (u(0), u_t(0)) \|_{B_{d/2-1}^{d/2-1} \times B_{d/2-2}^{d/2-2}} + \sum \| \Omega_{ij} u(0) \|_{B_{d/2-1}^{d/2-1}} + \sum \| \Omega_{ij} u_t(0) \|_{B_{d/2-2}^{d/2-2}}.
\]

\(^a\) for the more general Yang-Mills system.

\(^b\) Here \(\Omega_{ij} = x_i \partial_j - x_j \partial_i\) are the generators of the angular derivatives.
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In particular, this result shows global regularity for radial data, which is small in the critical Besov spaces.

The goal of this paper is to show global regularity result in $d \geq 5$ for the problem

$$\square u = \mathcal{P}(u \cdot \nabla u) + \mathcal{P}(u^3) \quad (t, x) \in \mathbb{R}^{1+d}.$$  \hfill (1.3)

Here $\mathcal{P}$ is the Leray projection onto divergence free vector fields. Clearly, by the form of the Yang-Mills system (1.1), this would imply global regularity for (1.1) as well.

More precisely, we assert that if $\sup_{-1/2 \leq \alpha \leq 1/2} \|(u(0), u_t(0))\|_{\dot{B}^{d/2+\alpha-1}_2 \times \dot{B}^{d/2+\alpha-2}_4} \ll 1$, one has a global solution for (1.3) and moreover the solution map is Lipschitz, see Theorem 1.1 below.

Next, we describe the heuristics of our approach, after which we will present an exact formulation of the main result. Let us start by pointing out, that all the estimates will be performed in the standard Strichartz spaces. That is, no $X^{s,b}$ type spaces will be utilized. The difficulty one faces with quadratic wave equations problems like (1.3) in that approach is the restricted range of the Strichartz estimates. In fact, disregarding the required favorable grouping of the derivatives, we are essentially required to place the nonlinearity $u \cdot \nabla u \in L^1_t L^2_x$. But by Hölder’s inequality, that would mean placing $|\nabla|^\alpha u \in L^2_t L^4_x$ for appropriate $\alpha$. Even though $L^2_t L^4_x(\mathbb{R}^d)$ is (barely) available for $d \geq 5$, it turns out that in any dimension, one cannot organize the derivatives to fall into place. We must mention that the argument above may be enhanced considerably, if one uses the powerful bilinear estimates of Wolf and Tao. Such an approach will inevitably involve the $X^{s,b}$ spaces machinery, which in turn will force one to address the resulting logarithmic divergences. The author however is not aware of any reference, where such an approach has been attempted.

Our method is in a way very similar to the one employed by Shatah, [14] and it consists of changing variables, so that the quadratic system (1.3) becomes a cubic one, which certainly gives much more flexibility with the Strichartz range. Let us describe the idea behind this change of variables. We find a (singular) bilinear operator $\Lambda(u, v) : \square \Lambda(u, u) = \mathcal{P}(u \cdot \nabla u) + O(u^3)$ (in doing so, recall that $\square u \sim u\partial u + F(u) = O(u^3)$). It is then clear that the difference $v := u - \Lambda(u, u)$ satisfies a non-linear wave equation with a cubic nonlinearity, which allows us to make the argument.

Let us introduce the function spaces in which we will perform our global regularity estimates. For any wave admissible pair $(q, r)$ (see Section 2.3 for a precise definition), define the space of divergence-free functions with the norm

$$\|u\|_{X^{s}} := \sup_{q, r - \text{wave adm.}} \sum_{k} 2^{k(1/q + d/r + s - 1)} \|u_k\|_{L^q_t L^r_x}. $$

which is a serious issue by itself, but really secondary to the lack of enough Strichartz estimates!
Note that for all such pairs \((q,r)\), we have \(\|\nabla \|^1/q + d/r - 1\) \(u\|_{L^q_t L^r_x} \lesssim \|u\|_X\) and hence \(X^0\) is embedded in the standard scale-invariant Strichartz space for this problem.

**Theorem 1.1.** For any \(d \geq 5\), there exists \(\varepsilon = \varepsilon(d)\), so that whenever \(f,g\) are divergence free and
\[
(\|f\|_{\dot{B}^{d/2-\alpha-1}} + \|g\|_{\dot{B}^{d/2-\alpha-2}}) < \varepsilon,
\]
then the system (1.3) with initial data \(u(0) = f, u_t(0) = g\) has an unique global solution, which satisfies
\[
\|u\|_{X^s} \leq C_d\|(f,g)\|_{\dot{B}^{d/2-\alpha-1} \times \dot{B}^{d/2-\alpha-2}}
\]
for some constant \(C_d\). Moreover, the solution map is Lipschitz. That is, there exists a constant \(C_d\), so that whenever for \((f_j,g_j)\) have small norms in \(\dot{B}^{d/2-1/2+\delta} \cap \dot{B}^{d/2-1/2-\delta}\) and \((f_j,g_j) \in \dot{B}^{d/2+\delta-1} \times \dot{B}^{d/2+\delta-2}\), the corresponding solutions \(u_j : u_j(0) = f_j, \partial_t u_j(0) = g_j\) satisfy
\[
\|u_1 - u_2\|_{X^0} \leq C_d\|(f_1 - f_2, g_1 - g_2)\|_{\dot{B}^{d/2+\delta-1} \times \dot{B}^{d/2+\delta-2}}.
\]

**Remarks:**
- One could further squeeze the solution space \(X^s\) by adding the norms
\[
\sum_k 2^k (d/2 + s - 2) \|\partial_t u_k\|_{L^\infty L^2}
\]
in the definition of \(X\). This would however necessitate the addition of some tedious computations and we chose not to include it.
- Regarding the Sobolev spaces global regularity in for the 4D MKG and in general for the 4 D Yang-Mills problem, it appears that the methods developed in this paper will certainly be useful, but insufficient.

We have the following immediate corollary of the somewhat technical formulation of Theorem 1.1, which is more in the spirit of the global regularity results.

**Corollary 1.2.** For \(d \geq 5, \delta > 0\), there exists \(\varepsilon_0 = \varepsilon_0(d,\delta)\), so that whenever
\[
0 < \varepsilon < \varepsilon_0
\]
\[
\|(f,g)\|_{H^{d/2-1/2+\delta} \times H^{d/2-1/2-\delta}} < 1
\]
then the system \(\Box \psi = F[u \cdot \nabla u] + F[u^3]\) with Cauchy data \(u(0) = \varepsilon f, u_t(0) = \varepsilon g\) has an unique global solution \(\|u\|_{L^\infty(0,\infty) H^{d/2-1/2+\delta}} < \infty\).

2. Preliminaries

We start by introducing some basic concepts in Fourier analysis.
2.1. Littlewood-Paley operators

The Fourier transform and its inverse are defined via
\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx, \\
f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix\cdot\xi} d\xi.
\]

For a positive, smooth and even function \( \chi : \mathbb{R}^1 \to \mathbb{R}_+^1 \), supported in \( \{ \xi : |\xi| \leq 2 \} \) and so that \( \chi(\xi) = 1 \) for all \( |\xi| \leq 1 \). Define \( \varphi(\xi) = \chi(\xi) - \chi(2\xi) \), which is supported in the annulus \( 1/2 \leq |\xi| \leq 2 \). Clearly \( \sum_{k \in \mathbb{Z}} \varphi(2^{-k} \xi) = 1 \) for all \( \xi \neq 0 \).

The \( k^{th} \) Littlewood-Paley projection is given by \( \tilde{S}_k f(\xi) = \varphi(2^{-k} \xi) \hat{f}(\xi) \). Note that the kernel of \( P_k \) is integrable uniformly in \( k \) and thus \( S_k : L^p \to L^p \) for \( 1 \leq p \leq \infty \) and \( \|S_k\|_{L^p \to L^p} \leq C_\nu \|\chi\|_{L^1} \). In particular, the bounds are independent of \( k \). Oftentimes, we will use \( f_k := S_k f \).

We will also need the Lorentz spaces \( L^{p,q} \) defined for example by the real interpolation formula \( L^{p,2} = (L^p, L^r)_{(\theta,2)} \), where \( q \neq r, \theta \in (0,1) : 1/p = \theta/r + (1-\theta)/q \).

2.2. Special partitions of unity

In this section, we introduce a partition of unity, with special localization properties of its Littlewood-Paley pieces. This will be needed in the sequel and it is just a technical preparation.

Lemma 2.1. Let \( d \geq 3 \). For every integer \( N \), there exists a function \( \psi = \psi^N : \mathbb{R}^d \to \mathbb{R}^1 \) and a constant \( C_{N,d} \), so that

- \( \psi(z) = \zeta(z) - \zeta(2z) \), \( \zeta = \zeta_{<0} \in \mathcal{S} \). As a consequence
  \[
  \sum_{l=-\infty}^{\infty} \psi(2^l z) = 1, \quad z \neq 0,
  \]

- \( \zeta(0) = 1, \frac{\partial^\alpha}{\partial z^\alpha} \zeta(0) = 0 \) for all \( |\alpha| < N \)

- The vector-valued function \( \phi(z) := \psi(z)|z|^{-2} \) is a Schwartz function and it satisfies for all multi-indices \( \beta \) and \( m > 5 \),
  \[
  |\partial^\beta \phi_m(A)| \leq 2^{|\beta|} N_2^{|\beta|+1} 2^{-mN} \leq A > -N \quad (2.1)
  \]
  \[
  |\partial^\beta \phi_{<0}(A)| \leq C_{N,d}^{-|\beta|+1} < A > -N \quad (2.2)
  \]

Proof. The first few lines of the statements are just definitions, that is take a Schwartz function, \( \zeta = \zeta_{<0} \), so that \( \zeta(0) = 1, \frac{\partial^\alpha}{\partial z^\alpha} \zeta(0) = 0 \) for all \( |\alpha| < N \). Clearly \( \sum_l \psi(2^l z) = 1 \) for all \( z \neq 0 \) and \( \frac{\partial^\alpha}{\partial z^\alpha} \psi(0) = 0 \). Since \( \psi \) is (real) analytic and \( \psi(0) = 0 \), it follows that \( \phi(z) \in C^\infty \), which in addition to the decay properties of \( \psi \), implies that \( \phi \) is a Schwartz function as well. While the property (2.2) is just a corollary of the fact that \( \phi \) is a Schwartz function (note that \( \psi(0) = 0 \), we still
need to show (2.1). We will show it for $\beta = 0$, the rest being similar. Represent $\Delta \phi(\xi) = c_d \nabla \phi(\xi)$, whence
\[
\phi_m(A) = c_d \int \frac{\varphi(2^{-m}\xi)}{|\xi - \eta|^{d-2}} \nabla \phi(\eta)e^{i\xi \cdot \eta} d\xi d\eta
\]
The estimate by $|A|^{-N}$ now follows, if we just integrate by parts $N$ times (note that we pick up in the process $2^{-mN}$). This however does not work if $|A| \ll 1$. To achieve the desired rate of decay $2^{-mN}$ even in the case $|A| \ll 1$, note first that the $\eta$ integration is over the ball $\{ \eta : |\eta| \leq 1 \}$ and write the Taylor expansion for the function $h(\eta) = |\xi - \eta|^{2-d}$,
\[
\frac{1}{|\xi - \eta|^{d-2}} = \sum_{\alpha:|\alpha| \leq N-1} \frac{[d^\alpha/d\eta^n]h(0)}{\alpha!} + O(|\eta|^N).
\]
Using this in the expression for $\phi_m(A)$ and taking into account $\int \eta^\alpha \nabla \phi(\eta) d\eta = 0$ for all $|\alpha| \leq N - 1$, we get an estimate
\[
|\phi_m(A)| \leq C_d \int |\varphi(2^{-m}\xi)||\xi|^{2-d-N} d\xi \int_{|\eta| < 1} |\nabla \phi(\eta)||\eta|^N d\eta \leq C_N d 2^{-m(N-2)}.
\]
as claimed. \hfill \Box

2.3. Strichartz estimates

For the wave equation
\[
\begin{align*}
\Box u &= F, \quad (t, x) \in \mathbb{R}^{1+d}, d \geq 3 \\
u(0) &= f, \ u_t(0) = g.
\end{align*}
\]
there are the well-known Strichartz estimates
\[
\|\nabla |^{1/q + d/r + \alpha} u \|_{L_t^q L_x^r} \lesssim \| (f, g) \|_{\dot{H}^{d/2 + \alpha} \times \dot{H}^{d/2 + \alpha - 1}} + \| \nabla |^{d/2 + \alpha - 1} F \|_{L_t^2 L_x^2}.
\]
(2.3)
for any real $\alpha$ and whenever $d (q, r)$ is wave admissible pair, i.e. $q, r \geq 2, 1/q + (d - 1)/(2r) \leq (d - 1)/4$. Due to the Littlewood-Paley theory and the inclusion $L^q \subset L^r$, for all $q < r$, one has the Besov form of (2.3)
\[
\sum_k 2^{k(1/q + d/r + \alpha)} \| u_k \|_{L_t^q L_x^r} \lesssim \| (f, g) \|_{\dot{B}^{d/2 + \alpha} \times \dot{B}^{d/2 + \alpha - 1}} + \sum_k 2^{k(d/2 + \alpha - 1)} \| F_k \|_{L_t^1 L_x^2},
\]
(2.4)
which will be useful in the sequel.

For the Dirac equation
\[
\partial_t U \pm i \nabla U = G(t, x),
\]
we have, for all wave admissible pairs $(q, r)$
\[
\| \nabla |^{1/q + d/r + \alpha} U \|_{L_t^q L_x^r} \lesssim \| U(0) \|_{\dot{H}^{d/2 + \alpha}} + \| \nabla |^{d/2 + \alpha} G \|_{L_t^1 L_x^2}.
\]
(2.5)
and the corresponding Besov versions like in (2.4).

\textsuperscript{4}In dimension three, one has to exclude the point $q = 2, r = \infty$, which otherwise satisfies the requirements.
2.4. Fourier restriction to spherical caps

For every positive integer \( l \), introduce also a family of unit vectors \( \{ \theta_j \}_j \), so that the balls \( B(\theta_j, 2^{-1}) \) have the finite intersection property with each point covered by no more than a fixed (depending only on dimension) number of balls \( C_d \). Introduce also a partition of unity subordinated to this cover, that is a family of \( C^\infty \) functions \( \{ \chi_{j,l} \} \), so that

\[
\sum_j \chi_{j,l}(2^j(\xi/|\xi| - \theta_j^l)) = 1.
\]

for every \( \xi \neq 0 \). This can be done in such a way that \( \sup_{j,l} |\partial^\gamma \chi_{j,l}(x)| \leq C_{d,\gamma} \).

The following lemma is standard and appears as Lemma 2, [19].

Lemma 2.2. Let \( \theta_0 \in S^{d-1} \) and \( \zeta \) is a \( C^\infty \) function with \( \text{supp} \, \zeta \subset \{ \xi : 1/2 \leq |\xi| \leq 2 \} \). Let also \( l > 0, k \) be pair of integers. Define \( K_{l,k,\theta_0} \) to be the inverse Fourier transform of \( \varphi(2^l(|\xi|/|\xi| - \theta_0))\zeta(2^{-k}|\xi|) \), that is

\[
K_{l,k,\theta_0}(x) = \int \zeta(2^l(|\xi|/|\xi| - \theta_0))\varphi(2^{-k}|\xi|)e^{2\pi ix \cdot \xi} d\xi.
\]

Then

\[
\sup_{\theta_0, l,k} \int |K_{l,k,\theta_0}(x)| dx \leq C_d \sup_{0<s<d+1} \sup_A |\partial^s \zeta(A)| \sup_{0<s<d+1} \sup_A |\partial^s \varphi(A)|. \tag{2.6}
\]

In particular,

\[
\|Z_{\theta_0,l,k} g(\xi) := \zeta(2^l(|\xi|/|\xi| - \theta_0))\varphi(2^{-k}|\xi|) \hat{g}(\xi)\|
\leq C_d \sup_{0<s<d+1} \sup_A |\partial^s \zeta(A)| \sup_{0<s<d+1} \sup_A |\partial^s \varphi(A)|. \tag{2.7}
\]

We would like to point out that in (2.7), the right hand side is independent of both \( l, k \). An immediate application of such a result is the following

Corollary 2.3. Let \( P_{j,l}^k \) be the operators restricting the Fourier transform to \( \{ \xi : |\xi| \sim 2^k, |\xi|/|\xi| - \theta_j^l \leq 2^{-1} \} \), that is \( \hat{P}_{j,l}^k f(\xi) := \chi_{j,l}(2^j(|\xi|/|\xi| - \theta_j^l))\varphi(2^{-k}|\xi|) \hat{f}(\xi) \). Let \( 1 < q < \infty \) and \( \tilde{q} := \text{max}(q, q') \). Then

\[
\left( \sum_j \| P_{j,l}^k f \|_{L^\tilde{q}}^q \right)^{1/\tilde{q}} \leq C \| f \|_{L^q} \tag{2.8}
\]

Also, for \( 2 \leq q < \infty \),

\[
\sum_j \| P_{j,l}^k f \|_{L^q} \leq C \left( \sum_j \| P_{j,l}^k f \|_{L^{q'}}^q \right)^{1/q'} \tag{2.9}
\]

Proof. The proof of such a result consists of observing that the statement follows by interpolation between the \( L^2 \) result and the \( L^p, 1 \leq p < \infty \) results. Indeed, the
$L^2$ estimate follows from Plancherel’s, while the $L^p$, $1 \leq p \leq \infty$ estimates follow from the uniform $L^1$ bounds for the kernel of $P^k_{j,l}$ from Lemma 2.2, whence for $1 \leq p \leq \infty$,

$$\sup_{j,k,l} \|P^k_{j,l} f\|_{L^p} \leq C_d \|f\|_{L^p}$$

Interpolating between the last estimate and the $L^2$ estimate yields (2.8).

As far as (2.9) goes, it follows again by interpolation between the endpoints $q = 2$ (recall the almost disjointness of the Fourier support) and $q = \infty$, which is a simple consequence of the triangle inequality. \hfill \Box

### 2.5. Bilinear multipliers

In this section, we show that a bilinear multiplier in the form $\zeta(\xi/|\xi|, \eta/|\eta|)$, where $\zeta \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ gives rise to a bounded multiplier $L^p \times L^q \to L^r$, whenever $1/r = 1/p + 1/q$, $1 < r, p, q < \infty$. Moreover, one has the estimate

$$\|\Xi\|_{L^p \times L^q \to L^r} \leq C_d \sup_{\alpha \in \mathbb{N}^{2d}, \|\alpha\| \leq 4d+4} \sup_{u,v} |\partial^{\alpha} \zeta(u,v)|$$

**Proof.** Note that we may assume without loss of generality that $\text{supp} \zeta$ is a compact subset of $\{|\xi| < 2\pi, |\eta| < 2\pi\}$, since otherwise we may multiply (without changing the operator $\Xi$) the multiplier $\zeta(\xi/|\xi|, \eta/|\eta|)$ by $\varphi(\xi/|\xi|)\varphi(\eta/|\eta|)$. Next, we expand the function $\zeta(u,v)$ in a double Fourier series

$$\zeta(u,v) = \sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} a_{nm} e^{i(n\cdot u + m\cdot v)}.$$  

This expansion is valid in particular for $u, v : |u| = 1 = |v|$ and the coefficients $a_{nm}$ decay faster than any polynomial. That is for every integer $N$, there is $C_N$, so that

$$|a_{nm}| \leq C_N \sup_{\alpha,\beta : |\alpha|, |\beta| \leq N} \sup_{u,v} |\partial^\alpha u^\beta \partial^\beta v^\alpha (\zeta(u,v))| (< n > + < m >)^{-N}.$$  

Using the Fourier expansion, we rewrite the operator as follows

$$\Xi(f, g) = \sum_{n,m \in \mathbb{Z}^d} a_{nm} \Xi_n(f) \Xi_m(g).$$

*In particular, one should note that such multipliers do not fit the standard conditions of [3].
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where $\Xi_n(f) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} \hat{f}(\xi) e^{i\xi \cdot \xi} d\xi$. The $L^p \to L^p$ norm of the operators $\Xi_n$ may be estimated (very roughly) by the Hörmander-Mikhlin’s theorem, which gives

$$\|\Xi_n\|_{L^p \to L^p} \leq C_{d,p} \sup_{|\alpha| \leq |d/2| + 1} \sup_{\eta} |\alpha| |\partial^{\alpha}_{\eta} (e^{i\eta \cdot \eta}/|\eta|)|.$$ 

That yields $\|\Xi_n\|_{L^p \to L^p} \leq C_{d,p} |n|^{(d/2) + 1}$, whence

$$\|\Xi(f,g)\|_{L^r} \leq C_{d,p,q} \sum_{n,m} |a_{nm}| \|\Xi_n(f)\|_{L^p} \|\Xi_m(g)\|_{L^q} \leq C_{N,d,p,q} \|f\|_{L^p} \|g\|_{L^q} \sum_{n,m} (|n| + |m|)^{-N} |n|^{d/2} |m|^{d/2} + 1$$

Clearly taking $N = 2d + 2$ yields the convergence of the series and hence the Lemma.

2.6. Bernstein inequality

**Lemma 2.5.** Let $Q \subset \mathbb{R}^d$ be any set. Let $P_Q$ denotes the operator $\hat{P}_Q f(\xi) := \chi_Q(\xi) \hat{f}(\xi)$ for some smooth function $\chi_Q, \text{supp}\chi_Q \subset Q$. Then, for any $1 \leq p \leq q \leq \infty$, one has

$$\|P_Q f\|_{L^q} \leq (C_d \|\chi_Q\|_{L^1}) |Q|^{1/p - 1/q} \|f\|_{L^p}$$

**Remark:** Usually in the applications, we have $\|\chi_Q\|_{L^1(\mathbb{R}^d)} \leq C_d$. For example, this holds for $\chi_Q(\xi) = \chi((\xi - \xi_Q)/d)$. In general, for any function $f$, Fourier supported on a set $A$, which is well approximated by rectangles (i.e. there exist rectangles $R_1 \subset A \subset R_2$, so that $|R_2| \leq C_d |R_2|$), we have $\|f\|_{L^q} \leq C_d |A|^{1/p - 1/q} \|f\|_{L^p}$.

3. Bilinear change of variables that reduces the problem to a cubic NLW equation

In this section, we shalldescribe a particular change of variables that helps us recast (1.3) as a nonlinear wave equation with cubic nonlinearities. Let us start our discussion with a few heuristics.

3.1. Some heuristics

If we run an iteration scheme for the problem $\square u = Z(u,u)$, (where $Z(u,v)$ is a bilinear form), we have the initial iteration step

$$u^0 = \cos(t|\nabla|) f + |\nabla|^{-1} \sin(t|\nabla|) g.$$ 

This of course does not take into account the nonlinearity, so let us continue a step further. That is, we solve

$$\square u^1 = Z(u^0, u^0).$$  \hspace{1cm} (3.1)
Note that on the right hand side, the entry \( u^0 : \Box u^0 = 0 \). Thus, it is clear that it will be beneficial to construct a bilinear operator, which solves (3.1) in the form
\[
\Lambda(u,v)(t,x) = (2\pi)^{-d-2} \int \sigma(\xi,\eta)Z(\hat{u}(\tau_1,\xi),\hat{v}(\tau_2,\eta))e^{\pi i(\xi+\eta)(x^+\tau_1^++x^-\tau_2^+)|t|}d\xi d\eta d\tau_1 d\tau_2
\]
In order to describe \( \Lambda \), one needs to take care of an additional technical detail, namely we need to fix the sign of \( \tau \). More specifically, introduce \( \tau^\pm(\tau,\xi) := \chi_{\tau>0}\hat{u}(\tau,\cdot), \tau^-\hat{u}(\tau,\cdot) \). Observe that \( u^\pm = (u \pm H_t u)/2 \), where \( H_t \) is the Hilbert transform in the time variable. Thus, by vector-valued Calderon-Zygmund theory, (see for example [1]) \( \|u^\pm\|_{L_t^q L_x^\infty} \lesssim \|u\|_{L_t^q L_x^\infty} \). Write now \( \Lambda(u,v) \) in the form
\[
\Lambda(u,v) = \Lambda^{++}(u^+,v^+) + \Lambda^{+-}(u^+,v^-) + \Lambda^{-+}(u^-,v^+) + \Lambda^{--}(u^-,v^-)
\]
How do we construct \( \Lambda^{++} \) (with symbol \( \sigma^{++} \))? We will simply require
\[
\Box \Lambda^{++}(u^+,v^+) = Z(u^+,v^+). \tag{3.2}
\]
Compute the symbol of \( \Box \Lambda^{++}(u^+,v^+) \)
\[
[(\tau_1 + \tau_2)^2 - |\xi + \eta|^2]^{\sigma^{++}}(\xi,\eta) = 2(|\xi||\eta| - \langle \xi, \eta \rangle)\sigma^{++}(\xi,\eta) =
\]
\[
|\xi||\eta|/|\xi| - |\eta|/|\eta|)^2 \sigma^{++}(\xi,\eta),
\]
where we have used repeatedly \( \tau_1 = |\xi|, \tau_2 = |\eta| \). Thus, in order to achieve (3.2), we need
\[
\sigma^{++}(\xi,\eta) = \frac{1}{|\xi||\eta|/|\xi| - |\eta|/|\eta|)^2}.
\]
Similarly, one can deduce the formula for \( \sigma^{\pm,\pm} \) for \( \varepsilon_1, \varepsilon_2 \) \( = \pm 1 \)
\[
\sigma^{\pm,\pm}(\xi,\eta) = \frac{1}{|\xi||\eta|/|\xi| - \varepsilon_1 \varepsilon_2 \eta/|\eta|)^2} \tag{3.3}
\]

### 3.2. The formal definition

Inspired by the heuristics presented in the previous section, we will define a change of variables via a symbol similar to (3.3).

To begin with, we need to define \( U^+, U^- \) similar to \( u^+, u^- \) before. Recall that for any \( u \) solving the initial value problem \( \Box u = N, u(0) = f, \partial_t u(0) = g \), there is the Duhamel’s formula
\[
u = \cos(t|\nabla|)f + |\nabla|^{-1} \sin(t|\nabla|)g + |\nabla|^{-1} \int_0^t \sin((t-s)|\nabla|)N(s,\cdot)ds.
\]
Let \( U^+, U^- : u = U^+ + U^- \)
\[
U^+ = \frac{1}{2} e^{it|\nabla|} f + \frac{1}{2i|\nabla|} e^{it|\nabla|} g + \frac{1}{2i|\nabla|} \int_0^t e^{i(t-s)|\nabla|} N(s,\cdot)ds.
\]
\[
U^- = \frac{1}{2} e^{-it|\nabla|} f - \frac{1}{2i|\nabla|} e^{-it|\nabla|} g - \frac{1}{2i|\nabla|} \int_0^t e^{-i(t-s)|\nabla|} N(s,\cdot)ds.
\]
\(^{\dagger}\)Recall that in the setting of (3.1) \( u^+, v^+ \) are simply solutions of the free wave equation.
By the definition of these two functions, for \( u : \square u = \mathcal{P}(u \cdot \nabla u) + \mathcal{P}[F(u)] \), we get
\[
\partial_t U^+ - i|\nabla|U^+ = \frac{1}{2i} |\nabla|^{-1} [\mathcal{P}(u \cdot \nabla u) + \mathcal{P}[F(u)]] \tag{3.4}
\]
\[
\partial_t U^- + i|\nabla|U^- = -\frac{1}{2i} |\nabla|^{-1} [\mathcal{P}(u \cdot \nabla u) + \mathcal{P}[F(u)]] \tag{3.5}
\]
That is, \( U^+ \) and \( U^- \) satisfy a Dirac equation with a right hand side, which is a multiple of \( |\nabla|^{-1} [\mathcal{P}(u \cdot \nabla u) + \mathcal{P}[F(u)]] \). Furthermore, to show \( u \in \mathcal{X} \), it will suffice to show \( U^+, U^- \in \mathcal{X} \), which allows us to work with \( U = (U^+, U^-) \) henceforth.

Since \( \|u\|_\mathcal{X} \lesssim \|U\|_\mathcal{X} \), our aim is to show estimate of \( \|U\|_\mathcal{X} \) in terms of \( \|\mathcal{P}(f,g)\|_{L^2_{t,x}} \) and powers of \( \|U\|_\mathcal{X} \). Write
\[
\mathcal{P}(u \cdot \nabla u) = \mathcal{P}(U^+ \cdot \nabla U^+) + \mathcal{P}(U^- \cdot \nabla U^-) + \mathcal{P}(U^+ \cdot \nabla U^-) + \mathcal{P}(U^- \cdot \nabla U^+),
\]
we will define a change of variables \( (U^+, U^-) \rightarrow (V^+, V^-) \)
\[
U^+ = V^+ + \frac{1}{2i} |\nabla|^{-1} \Omega^+(u, u) \tag{3.6}
\]
\[
U^- = V^- - \frac{1}{2i} |\nabla|^{-1} \Omega^-(u, u) \tag{3.7}
\]
\[
\Omega^\pm(u, u) := \sum_{\varepsilon_1, \varepsilon_2 = \pm} \Omega^{\varepsilon_1, \varepsilon_2}(U^{\varepsilon_1}, U^{\varepsilon_2}),
\]
and \( \Omega^{\pm: \pm: \pm} \) are to be introduced momentarily. For clarity, let us define first \( \Omega^{+: +: +} \), the general formula to follow shortly. Take
\[
\Omega^{+: +: +}(w, z) = -i(2\pi)^{-2d} \int \frac{\mathcal{P}(\xi + \eta)}{|\xi| + |\eta| - |\xi + \eta|} \langle \hat{w}(t, \xi, \eta), \hat{z}(t, \xi, \eta) \rangle e^{i(\xi + \eta) \cdot x} \, d\xi \, d\eta,
\]
where \( \mathcal{P}(\xi + \eta) \) denotes the symbol of the Leray projection. Let us verify first the action of the Dirac operator \( \partial_t - i|\nabla| \) on \( \Omega^{+: +: +} \). We have
\[
(\partial_t - i|\nabla|)\Omega^{+: +: +}(U^+, U^+)(t, x) =
\]
\[
= c_d \int \frac{(\tau_1 + \tau_2 - |\xi + \eta|)\mathcal{P}(\xi + \eta) \langle U^{\tau_1}(\tau_1, \xi, \eta) U^{\tau_2}(\tau_2, \eta) e^{i(\xi + \eta) \cdot (\tau_1 + \tau_2) t} \rangle}{|\xi + \eta| - |\xi + \eta|}
\]
\[
= \Omega^{+: +: +}((\partial_t - i|\nabla|)U^+, U^+) + \Omega^{+: +: +}(U^+, (\partial_t - i|\nabla|)U^+) + \mathcal{P}(U^+ \cdot \nabla U^+) =
\]
\[
= \mathcal{P}(U^+ \cdot \nabla U^+) + \frac{1}{2i} \Omega^{+: +: +}([\nabla]^{-1} \mathcal{P}(u \cdot \nabla u), U^+) +
\]
\[
+ \Omega^{+: +: +}(U^+, [\nabla]^{-1} \mathcal{P}(u \cdot \nabla u)) + \frac{1}{2i} \Omega^{+: +: +}([\nabla]^{-1} \mathcal{P}(F(u)), U^+) +
\]
\[
+ \Omega^{+: +: +}(U^+, [\nabla]^{-1} \mathcal{P}(F(u)))
\]
where the last equality follows from (3.4). It is clear now that similar calculations justify
\[
\Omega^{\varepsilon_1, \varepsilon_2}(w, z) = -i(2\pi)^{-2d} \int \frac{\mathcal{P}(\xi + \eta)}{|\xi| + |\eta| - |\xi + \eta|} \langle \hat{w}(t, \xi, \eta), \hat{z}(t, \xi, \eta) \rangle e^{i(\xi + \eta) \cdot x} \, d\xi \, d\eta.
\]
We start with the case of $\Omega^-$.  

4. The bilinear operators $T^+$ and $T^-$

In this section, we introduce a more tractable form of the operators $\Omega^\pm, \Omega^\pm$.

4.1. The operator $T^+$

We start with the case of $\Omega^{\pm, \pm, \pm}$, where $\varepsilon_1 \varepsilon_2 = 1$. There are several cases that arise, (one of them the trivial case $\varepsilon_1 = \varepsilon_2 = 1, \varepsilon_3 = -1$), but $\Omega^{+, +, +}$ is a representative one, so we work with it. Note that

$$\frac{1}{|\xi| + |\eta| - |\xi + \eta|} = \frac{|\xi| + |\eta| + |\xi + \eta|}{2(|\xi||\eta| - (\xi, \eta))} = \frac{|\xi| + |\eta| + |\xi + \eta|}{|\xi||\eta||\xi| - |\eta||\eta|}.$$  

(4.1)

Introduce the special partition of unity, discussed in Lemma 2.1, in the singular variable $\xi/|\xi| - \eta/|\eta|$. Namely,

$$\sum_{l=-\infty}^{\infty} \psi(2^l(\xi/|\xi| - \eta/|\eta|)) = 1,$$

for any pair of non-zero vectors $\xi, \eta \in \mathbb{R}^d$. Inserting this in the formula for $\Omega^{+, +, +}$ yields

$$\Omega^{+, +, +}(u, v)(x) = -i(2\pi)^{-2d} \int \frac{\mathcal{P}(\xi + \eta)}{|\xi| + |\eta| - |\xi + \eta|} \langle \hat{u}(\xi), \hat{v}(\eta) \rangle e^{i(\xi + \eta) \cdot x} d\xi d\eta =$$

$$= -i(2\pi)^{-2d} \int \frac{|\xi| + |\eta| + |\xi + \eta|}{|\xi||\eta||\xi| - |\eta||\eta||\xi|} \langle \hat{u}(\xi), \hat{v}(\eta) \rangle e^{i(\xi + \eta) \cdot x} d\xi d\eta =$$

$$= C_d \sum_{l=-\infty}^{\infty} \int \frac{\psi(2^l(\xi/|\xi| - \eta/|\eta|))(\xi|\xi| + |\xi| + |\xi + \eta|)\mathcal{P}(\xi + \eta)}{|\xi||\eta||\xi| - |\eta||\eta||\xi|} \langle \hat{u}(\xi), \hat{v}(\eta) \rangle e^{i(\xi + \eta) \cdot x} d\xi d\eta$$

Next, recall that in our model, we assume that the entries are divergence free vectors and hence $\langle \hat{u}(\xi), \xi \rangle = 0$. One can write

$$\psi(2^l(\xi/|\xi| - \eta/|\eta|))(\xi/|\xi| + |\xi| + |\xi + \eta|) = \psi(2^l(\xi/|\xi| - \eta/|\eta|)) \cdot |\xi|^{-1} \hat{u}(\xi).$$
where \( \phi(z) = \psi(z)z|z|^{-2} \) is the vector-valued Schwartz function introduced in Lemma 2.1. It is clear that the \( L^q \times L^r \to L^p \) mapping properties of \( \Omega^{+,+} \) are related to the \( L^q \times L^r \to L^p \) mapping properties of the following singular bilinear operator:

\[
T^+(f, g)(x) = \sum_{l=-\infty}^{\infty} 2^l \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(2^l(\xi/|\xi| - \eta/|\eta|)) \hat{f}(\xi) \hat{g}(\eta) e^{i(\xi + \eta) \cdot x} d\xi d\eta \tag{4.2}
\]

via the relation

\[
\Omega^{+,+}(u, v) = \mathcal{P}[T^+(u, v) + T^+(|\nabla|^{-1}u, |\nabla|v) + |\nabla|T^+(|\nabla|^{-1}u, v)]. \tag{4.3}
\]

As we will see, the operator \( T^+ \) has very nice properties as a bilinear operator. In particular, the structure of the symbol allows for a decomposition with good angular separation, which is convenient for \( L^2 \) type analysis, see also the discussion at the end of Section 4.2.

**4.2. The operator \( T^- \)**.

In this section, we consider an operator, which allows us to treat \( \Omega^{\varepsilon_1;\varepsilon_2} \), when \( \varepsilon_1 \varepsilon_2 = -1 \). As it turns out, by following the steps in the previous section, one can represent \( T^- \) in a form similar to (4.2), that is with symbol

\[
\sum_{l} 2^l \phi(2^l(\xi/|\xi| + \eta/|\eta|)), \quad \text{but this representation is insufficient to obtain the desired estimates.}
\]

That is, by doing so one would have lost some important cancellation properties of the operator. The reason is that the angular separation can be achieved in each of the entries \( u, v \) of \( T^-(u, v) \), but not for the output \( T^-(u, v) \), as is the case with \( T^+ \). Thus, we have to take a closer look at the properties of \( T^- \). Consider a representative bilinear form to fix the ideas, say \( \Omega^{+,+,-} \).

We represent the symbol in two ways

\[
\frac{1}{|\xi| - |\eta| - |\xi + \eta|} = \frac{|\xi| - |\eta| + |\xi + \eta|}{2(|\xi||\eta| + \langle \xi, \eta \rangle)} = -\frac{|\xi| - |\eta| + |\xi + \eta|}{|\xi||\eta||\xi/|\xi| + \eta/|\eta||^2} \tag{4.4}
\]

\[
\frac{1}{|\xi| - |\eta| - |\xi + \eta|} = -\frac{|\xi| + |\eta| - |\xi + \eta|}{|\xi||\xi + \eta||\xi + \eta/|\xi + \eta|| - \xi/|\xi||^2} \tag{4.5}
\]

As before, the problematic terms are \( |\xi/|\xi| + \eta/|\eta||^{-2} \) and \( |\xi + \eta/|\xi + \eta|| - \xi/|\xi||^{-2} \). Introduce the two special partitions of unity

\[
\sum_{l_1=-\infty}^{\infty} \psi(2^{l_1}(\xi/|\xi| + \eta/|\eta|)) = 1 = \sum_{l_2=-\infty}^{\infty} \psi(2^{l_2}((\xi + \eta)/|\xi + \eta| - \xi/|\xi|))
\]

\(^8\)We have intentionally stripped the \( t \) dependence from the definition of \( T \), since it manifests itself only through the functions \( u, v \).
Thus, write \( \sum_{l_x, l_y} = \sum_l \sum_{l_x \geq l} + \sum_l \sum_{l_y \geq l+1} \) and since \( \psi(z) = \zeta(z) - \zeta(2z) \),

\[
1 = \sum_l \psi(2^l(|\xi| + |\eta|/|\eta|))\zeta(2^l((\xi + \eta)/|\xi + \eta| - \xi/|\xi|)) + \\
+ \sum_l \psi(2^l((\xi + \eta)/|\xi + \eta| - \xi/|\xi|))\zeta(2^{l+1}((\xi + \eta)/|\xi + \eta| - \xi/|\xi|))
\]

We insert this partition of unity into the bilinear operator. We arrive at

\[
\Omega^{+:+,-}(u, v)(t, x) = \\
= \sum_{l=-\infty}^{\infty} \int \psi(2^l(|\xi| + |\eta|/|\eta|))\zeta(2^l((\xi + \eta)/|\xi + \eta| - \eta/|\eta|))\mathcal{P}(\xi + \eta) \\
\times \frac{(|\xi| - |\eta| + |\xi + \eta|)}{|\xi||\eta||\xi/|\xi| + |\eta/|\eta|^2}(\hat{u}(t, \xi), \eta)\hat{v}(t, \eta)e^{i(\xi + \eta)x}d\xi d\eta + \\
+ \sum_{l=-\infty}^{\infty} \int \psi(2^l((\xi + \eta)/|\xi + \eta| - \xi/|\xi|))\zeta(2^{l+1}((\xi + \eta)/|\xi + \eta| - \eta/|\eta|))\mathcal{P}(\xi + \eta) \\
\times \frac{(|\xi| + |\eta| - |\xi + \eta|)}{|\xi||\eta||\xi/|\xi| + |\eta/|\eta|^2}(\hat{u}(t, \xi), \eta)\hat{v}(t, \eta)e^{i(\xi + \eta)x}d\xi d\eta
\]

In the first sum above, write

\[
\frac{\langle \hat{u}(t, \xi), \eta \rangle}{|\xi||\eta||\xi/|\xi| + |\eta/|\eta|^2} = \frac{\langle |\xi|^{-1}\hat{u}(t, \xi), \eta/|\eta| + \xi/|\xi| \rangle}{|\xi/|\xi| + |\eta/|\eta|^2}
\]

while in the second sum, it is beneficial to represent

\[
\frac{\langle \hat{u}(t, \xi), \eta \rangle}{|\xi||\eta||\xi/|\xi| + |\eta/|\eta|^2} = \frac{\langle |\xi|^{-1}\hat{u}(t, \xi), \frac{\xi + \eta}{|\xi| + |\eta|} - \frac{\xi}{|\xi|} \rangle}{|\xi/|\xi| + |\eta/|\eta|^2}
\]

Note that in both formulas, the property \( \text{div}(u) = 0 \) (or rather its equivalent \( \langle \hat{u}(t, \xi), \xi \rangle = 0 \)) was used in a crucial way. The end result is that we have shown that the multiplier is still singular, but with a milder singularity of the form \( |\xi/|\xi| + |\eta/|\eta|^2 \) or \( |(\xi + \eta)/|\xi + \eta| - \xi/|\xi|\}^{-1} \) respectively. Define

\[
T^-(f, g) = \\
= \sum_l 2^l \int \phi(2^l(|\xi| + |\eta|/|\eta|))\zeta(2^l((\xi + \eta)/|\xi + \eta| - \xi/|\xi|))\hat{f}(\xi)\hat{g}(\eta)e^{i(\xi + \eta)x}d\xi d\eta + \\
+ \sum_l 2^l \int \phi(2^l((\xi + \eta)/|\xi + \eta| - \xi/|\xi|))\zeta(2^{l+1}((\xi + \eta)/|\xi + \eta| + |\eta/|\eta|))\hat{f}(\xi)\hat{g}(\eta)e^{i(\xi + \eta)x}d\xi d\eta.
\]

Our considerations now show that we have a formula analogous to (4.3), namely

\[
\Omega^{+:+,-}(u, v) = \mathcal{P}[T^- (u, v) \pm T^- (|\nabla|^{-1}u, |\nabla|v) \pm |\nabla| T^- (|\nabla|^{-1}u, v)]
\]

In general, for any configuration of \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \), we get

\[
\Omega^{\varepsilon_1, \varepsilon_2, \varepsilon_3}(u, v) = \mathcal{P}[\pm T^- (u, v) \pm T^- (|\nabla|^{-1}u, |\nabla|v) \pm |\nabla| T^- (|\nabla|^{-1}u, v)] \quad (4.6)
\]
4.3. Mapping properties of the bilinear operator $T$

We study the $L^q \times L^r \rightarrow L^p$ mapping properties of $T$. The main theorem of this section, which describes the relevant bilinear estimates is stated for frequency localized functions, which is the most appropriate setup, since we are dealing with Besov spaces.

**Theorem 4.1.** Let $T$ be either $T^+$ or $T^-$. Let $d \geq 2$, $1 < q, r < \infty$, so that $1/q + 1/r > 1/2 + 1/(d - 1)$. We also require that, if $\min(q, r) < 2$, then $\max(q, r) < 2(d - 1)$. Then, for every two integers $n, k : k \leq n + 3$,

$$\|T(f_k, g_n)\|_{L^2} \leq C_d 2^{kd\alpha(d,q,r)} \|f_k\|_{L^q} 2^{nd(1/q + 1/r - 1/2 - \alpha(d,q,r))} \|g_n\|_{L^r} \quad (4.7)$$

where $\alpha(d,q,r)$ is given by

$$\alpha(d,q,r) = \begin{cases} 1/q & r < 2 \\ 1/q + 1/r - 1/2 & r \geq 2 \end{cases}$$

Moreover, on the sharp line of admissibility, $1/q + 1/r = 1/2 + 1/(d - 1)$, we have

$$\|T(f_k, g_n)\|_{L^2} \leq C_d 2^{kd\alpha(d,q,r)} \|f_k\|_{L^q} 2^{nd(1/q + 1/r - 1/2 - \alpha(d,q,r))} \|g_n\|_{L^r} \quad (4.8)$$

In the high-high interaction case, i.e. for the operator $S_k[T(f_{\sim n}, g_n)]$, where $k \leq n + 3$, we have the estimate

$$\|S_k[T(f_{\sim n}, g_n)]\|_{L^2} \leq C_d 2^{kd\alpha(d,q,r)} \|f_{\sim n}\|_{L^q} 2^{nd(1/q + 1/r - 1/2 - \alpha(d,q,r))} \|g_n\|_{L^r} \quad (4.9)$$

for all $0 \leq \delta < 1/(d - 1)$ and $1/q + 1/r = 1/2 + 1/(d - 1)$.

**Remarks:**

- One may of course state the Theorem 4.1 in the corresponding Besov spaces and even Sobolev spaces by using the standard inclusions between them. We will not do so here, since we only need them in the form presented here.
- We strongly believe that Theorem 2 is sharp in the range of exponents $q, r$, but we have not made an effort to justify this on the obvious candidates for counterexamples. On the other hand, we do not know whether the restriction $\max(q, r) < 2(d - 1)$ in the case $\min(q, r) < 2$ is needed or not.
- One could also prove estimates for $T$ with target $L^p, p \neq 2$ spaces, but the lack of orthogonality forces slightly weaker results. We need and prove estimates like that, see Section 9.

Let us dispose first of the easy case, which occurs when the summation in the formulas for $T^+, T^-$ is over, say $l < 10$. In that case, $T^\pm$ is homogeneous of order zero bilinear operator with $C^\infty(S^{d-1})$ symbol. According to Lemma 2.4, we have that $T^\pm : L^q \times L^r \rightarrow L^p, 1/p = 1/q + 1/r$. In particular, for $\hat{q} : 1/2 = 1/r + 1/\hat{q}$ and by Sobolev embedding

$$\|T(f_k, g_n)\|_{L^2} \leq C_d \|f_k\|_{L^\hat{q}} \|g_n\|_{L^r} \leq C_d 2^{kd(1/q + 1/r - 1/2)} \|f_k\|_{L^q} \|g_n\|_{L^r}$$
which implies (4.7) and (4.8). Therefore, we shall henceforth assume that the sums in the definitions of $T^\pm$ are only over $l \geq 10$.

Our plan is as follows. First, we present the proof of Theorem 4.1. This is done in two steps, in Section 5, we establish a representation formula for $T^+$ and $T^-$ and then in Section 6 we use the this representation and Fourier orthogonality arguments to deduce Theorem 4.1. This part of the paper is self-contained and independent of the rest of the argument, so the reader interested in the proof of Theorem 1.1 based on Theorem 4.1, may skip their proofs and go directly to Section 7. There, we show how one can reduce Theorem 1.1 to some technical estimates involving $T$ in the spirit of Theorem 4.1. The last two sections are then devoted to the proofs of these (multi-linear) estimates, which are based on relatively standard methods of Littlewood-Paley theory and Theorem 4.1.

5. Representation formula for $T(f, g)$

We consider the operator $T = T^+$ first, the corresponding formula for the case $T = T^-$ requires a slight modification, which is indicated at the end of the Section. Write

$$\phi = S_{<10}\phi + \sum_{m=10}^\infty S_m[\phi] = \phi_{<10} + \sum_{m=10}^\infty \phi_m$$

For each fixed $l, m \geq 10$, introduce a partition of unity $\sum j \chi_{j,l+m}(2^l(\xi/|\xi| - \eta/|\eta|) - \theta_j^{l+m}) = 1$. That is on the support of each term, we have $|\eta/|\eta| - \theta_j^{l+m}| \lesssim 2^{-l-m}$. Entering this in the formula for $T^+$ yields

$$T^+(f, g) = \sum_{l \geq 10} 2^l \sum_j \int \phi_{<0}(2^l(\xi/|\xi| - \eta/|\eta|)) \chi_{j,l}(2^l(\eta/|\eta| - \theta_j^{l+m})) \hat{f}(\xi) \hat{g}(\eta) e^{i(\xi+\eta)x}$$

$$+ \sum_{l, m \geq 10} 2^l \sum_j \int \phi_m(2^l(\xi/|\xi| - \eta/|\eta|)) \chi_{j,l+m}(2^l+m(\eta/|\eta| - \theta_j^{l+m})) \hat{f}(\xi) \hat{g}(\eta) e^{i(\xi+\eta)x}$$

Note that in both of the sums above, the $j$ index runs over the net $\{\theta_j^{l+m}\}$. The next step is to expand the function $\phi_m(2^l(\xi/|\xi| - \eta/|\eta|))$ around the point $2^l(\xi/|\xi| - \theta_j^{l+m})$. We have by the Taylor's formula

$$\phi_m(2^l(\xi/|\xi| - \eta/|\eta|)) = \sum_{|\alpha| \geq 0} \phi_m^{(\alpha)}(2^l(\xi/|\xi| - \theta_j^{l+m}))(2^l(\eta/|\eta| - \theta_j^{l+m}))^\alpha \alpha!$$

$$= \sum_{\alpha \in Z^d: |\alpha| \geq 0} (-1)^{|\alpha|}(\alpha!)^{-1}2^{-m|\alpha|}\phi_m^{(\alpha)}(2^l(\xi/|\xi| - \theta_j^{l+m}))[(2^l+m(\eta/|\eta| - \theta_j^{l+m}))^\alpha],$$

The convergences of the Taylor series has to be of course justified!
which we will show converges, whenever \( \eta \) is in the set \( \{ \eta : |\eta/|\eta| - \theta_j|^{l+m} | \leq 2^{-l-m} \} \).

Indeed, by Lemma 2.1

\[
2^{-m|a|} \sup_z |\phi_m^{(a)}(z)| \leq C_d^{[a]},
\]

we have that the Taylor series is majorized term-wise by \( \sum_{\alpha \in \mathbb{Z}^d} C_d^{[a]} (\alpha!)^{-1} = e^{dC_d} < \infty \).

This allows us to represent

\[
T^+(f,g) = \sum_{l,m \geq 10} 2^l \sum_{\alpha \in \mathbb{Z}^d} (-1)^{|a|} (\alpha!)^{-1} \int 2^{-m|a|} \phi_m^{(a)}(2^l(|\xi| - \theta_j)^{l+m})) \hat{f}(\xi) e^{i\xi x} d\xi
\]

\[
\times \int \chi_j, l+m(2^l m(\eta/|\eta| - \theta_j^{l+m})) \hat{g}(\eta) e^{imx} d\eta =
\]

\[
= \sum_{\alpha \in \mathbb{Z}^d} (-1)^{|a|} (\alpha!)^{-1} 2^l \sum_{l,m \geq 10} [\tilde{P}_j^{l,m,a} f] [P_j, l+m g],
\]

where

\[
\tilde{P}_j^{l,m,a} f(\xi) := 2^{-m|a|} \phi_m^{(a)}(2^l(|\xi| - \theta_j^{l+m})) \hat{f}(\xi)
\]

\[
P_j, l+m g(\eta) = \chi_j, l+m(2^l m(\eta/|\eta| - \theta_j^{l+m})) \hat{g}(\eta).
\]

For the proofs of (4.7) and (4.8), it will be enough to show

\[
\sum_l 2^l \| \sum_j [\tilde{P}_j^{l,m,a} f] P_j, l+m g \|_{L^2} \leq \frac{B_d^{[a] + 1} \sum_{l,m \geq 10} 2^l m(\eta/|\eta| - \theta_j^{l+m}) \| f_k \|_{L^{r+2}} \times}
\]

\[
2^{d(1/q + 1/r - 1/2 - \alpha(d,q,r))} \| g_n \|_{L^{r,2}}.
\]

which is what we concentrate on. Denote for convenience

\[
T^+_i(f,g) = \sum_j \tilde{P}_j^{l,m,a} f P_j, l+m g.
\]

Lemma 2.1 is very suggestive in the way we should look at the operator \( \tilde{P}_j^{l,m,a} \).

Namely, the principal action of \( \tilde{P}_j^{l,m,a} \) is to restrict the Fourier transform to the cone \( \{ \eta : |\eta/|\eta| - \theta_j|^{l+m} | \leq 2^{-l} \} \). This is due to the almost exponential decay in \( 2^l (\eta/|\eta| - \theta_j|^{l+m}) \). Note that while on one hand there is certainly a significant overlap between these cones\(^k\), there is the decay factor \( 2^{-mN} \) in (2.1) to compensate for that later in the estimates. Thus, the model operator to keep in mind is

\[
T^{+, toy}_i(f,g) = \sum_j P_j f P_j g.
\]

\(^1\)From now on, we will not explicitly write the somewhat different expression, when we deal with \( \phi_{<a} \), although one should be aware that this case arises.

\(^2\)To keep the notations simple, we omit the dependence on \( m, \alpha \) from our notations as they will be fixed right until the end.

\(^3\)as \( m >> 1 \) and since \( \{ \theta_j^{l+m} \} \) forms a \( 2^{-l-m} \) net, each point on the sphere belongs to a number of cones, whose number is bounded above and below by a constant multiple of \( 2^m(d-1) \).
where \( P_j \) restricts to the cone \( \{ \xi : |\xi| - \theta_j \leq 2^{-l} \} \) and the estimate one needs to establish is
\[
\sum_{l>0} 2^l \| T_{l,+}^{+,toy}(f_k, g_n) \|_{L^2} \lesssim 2^{kd(1/r - 1/2)} \| f_k \|_{L^q} \| g \|_{L^r}.
\] (5.5)

In order to make the above heuristic arguments precise and to take care of the full operator, introduce a new partition of unity
\[
\chi(2^j(\xi/|\xi|) - \theta_j) + \sum_{h=1}^{\infty} \varphi(2^{j-h}(\xi/|\xi|) - \theta_j^+ + m) = 1
\]
in \( \hat{P}_l^{j,m,\alpha} f \). Introduce the operators
\[
\hat{Q}_{j,l,m,h} \hat{g}(\xi) := \varphi(2^{j-h}(\xi/|\xi|) - \theta_j^+ + m) \hat{g}(\xi),
\]
\[
\hat{Q}_{j,l,m} \hat{g}(\xi) := \chi(2^j(\xi/|\xi|) - \theta_j) \hat{g}(\xi),
\]
which restrict smoothly the Fourier transform to the set \( \{ \xi : |\xi| - \theta_j^+ + m \sim 2^{-l} 2^h \} \), when \( h \geq 1 \) and to the set \( \{ \xi : |\xi| - \theta_j^+ + m \lesssim 2^{-l} \} \) for \( h = 0 \). Thus, we may write
\[
\hat{P}_l^{j,m,\alpha} = \sum_{h=0}^{\infty} \hat{P}_l^{j,m,\alpha} \hat{Q}_{j,l,m,h}
\]
Note that
\[
T_{l,+}(f, g) = \sum_j \hat{P}_l^{j,m,\alpha} f P_{j,l+m} g = \sum_{h=0}^{\infty} \sum_{j} \hat{P}_l^{j,m,\alpha} \hat{Q}_{j,l,m,h} f ] [ P_{j,l+m} g ]
\]
and since both entries \( P_{j,l+m} g \) and \( \hat{P}_l^{j,m,\alpha} \hat{Q}_{j,l,m,h} f \) are Fourier supported inside the cone \( \{ \xi : |\xi/|\xi| - \theta_j^+ + m \lesssim 2^{-l} 2^h \} \), then so is their product. Therefore,
\[
T_{l,+}(f, g) = \sum_{h=0}^{\infty} \sum_{j} \hat{Q}_{j,l,m,h} \left( \hat{P}_l^{j,m,\alpha} \hat{Q}_{j,l,m,h} f ] [ P_{j,l+m} g ] \right),
\] (5.6)
where
\[
\hat{Q}_{j,l,m,h} \hat{g}(\xi) = \chi(2^{j-h} - 10(\xi/|\xi| - \theta_j^+ m)) \hat{g}(\xi).
\]
Formulas (5.6) and its toy version (5.4), which already contains all the important features of (5.6), will be our starting point for investigating the boundedness of the operator \( T^+ \) in various function spaces.

For the operator \( T^- \), it is clear that one can perform the same manipulations as for the \( T^+ \). The difference (which is mainly technical, not a conceptual one) is that there are two multipliers that need to be expanded in Taylor series. All in all, recalling the exact formula for \( T^- \), one gets the following representation for the \( l^{th} \) ‘toy operator’
\[
T_{l,-}^{+,toy}(f, g) = \sum_j P_j [ P_j f Z_j g ],
\] (5.7)
where \( P_j \) is a Fourier restriction operator to \( \{ \xi : |\xi/|\xi| - \theta_j \leq 2^{-l} \} \) with smooth multiplier and \( Z_j \) restricts (smoothly) to \( \{ \eta : |\eta/|\eta| + \theta_j \leq 2^{-l} \}. \)
Global regularity for Yang-Mills fields in $\mathbb{R}^{1+5}$

6. Proof of Theorem 4.1

We are now prepared to give the full details of the proof of Theorem 4.1. To streamline our exposition, we consider the proof of (5.5) first. That is we will show Theorem 6.1.

Before we start with the concrete estimates, let us point out that it suffices to consider the case, when $1/q + 1/r = 1/2 + 1/(d-1)$, since the general case follows by Sobolev embedding. Indeed, for every pair $q, r : 1/q + 1/r > 1/2 + 1/(d-1)$, there exists $q_1, r_1 : q_1 \geq q, r_1 \geq r$, $1/q_1 + 1/r_1 = 1/2 + 1/(d-1)$, whence one can deduce (4.7) for $q, r$ from the corresponding one for $q_1, r_1$ via Sobolev embedding. From now on, without loss of generality, fix a pair $1 < q_1, r_1 < \infty : 1/q_1 + 1/r_1 = 1/2 + 1/(d-1)$.

6.1. Proof of (5.5)

By Fourier support considerations

$$T^+_{i,t,o}(f, g) = \sum_j P_j f P_j g = \sum_j P_j [P_j f P_j g].$$

This is because the product of two functions, both Fourier supported in a fixed one-sided cone is also supported in the same cone. Thus, we have achieved Fourier separation in $T^+_{i,t,o}$, whence

$$\|T^+_{i,t,o}(f_k, g_n)\|_{L^2(\mathbb{R}^d)}^2 \lesssim \sum_j \|P_j [P_j f_k P_j g_n]\|_{L^2}^2$$

whence by Hölder’s inequality and $\|P_j\|_{L^2 \to L^2} \lesssim 1$,

$$\|T^+_{i,t,o}(f_k, g_n)\|_{L^2(\mathbb{R}^d)}^2 \leq \left( \sum_j \|P_j f_k\|_{L^{p_1}}^{2s_1} \right)^{1/s_1} \left( \sum_j \|P_j g_n\|_{L^{p_2}}^{2s_2} \right)^{1/s_2}$$

(6.1)

for all $p_1, p_2, s_1, s_2 : 1/p_1 + 1/p_2 = 1/2, 1/s_1 + 1/s_2 = 1$. We would like to keep our options open and choose the most favorable configuration of $p_1, p_2, s_1, s_2$. This choice will depend on our fixed indices $q_1, r_1$.

Next, an application of the Bernstein inequality yields for every $q \leq p_1$ and $r \leq p_2$,

$$\|P_j f_k\|_{L^{p_1}} \leq C_d (2^{-(d-1)2kd})^{1/q-1/p_1} \|P_j f_k\|_{L^r}$$

$$\|P_j g_n\|_{L^{p_2}} \leq C_d (2^{-(d-1)2nd})^{1/r-1/p_2} \|P_j g_n\|_{L^r}$$

This follows by Lemma 2.5 and by computing the volume of the Fourier support of $P_j f_k$ to be $2^{-(d-1)2kd}$ and similarly for $P_j g_n$. Inserting the last two expressions into our previous estimates for $T^+_{i,t,o}$ yields

$$\|T^+_{i,t,o}(f_k, g_n)\|_{L^2(\mathbb{R}^d)} \leq C_d 2^{-d(1/q-1/p_1-1/p_2)} 2^{kd} (1/q-1/p_1) 2^{nd} (1/r-1/p_2) \times$$

$$\times (\sum_j \|P_j f_k\|_{L^r}^{2s_1})^{1/2s_1} (\sum_j \|P_j g_n\|_{L^r}^{2s_2})^{1/2s_2}$$
It is not hard to check that for all \((q, r)\) in a small neighborhood of \((q_1, r_1)\), one has a good choice of \(p_1, p_2, s_1, s_2\). Indeed, take one such pair \((q, r)\), say \(1/q + 1/r \in (1/2 + 1/(d-1) - \varepsilon, 1/2 + 1/(d-1) + \varepsilon)\) for some small \(\varepsilon\), to be determined below.

Choose \(s_1 : 2s_1 = \tilde{q} = \max(q, q')\). We will show that \(2s_2 \geq \tilde{r} = \max(r, r')\).

Indeed, all we have to show is that \(1/q + 1/r > 1/2\) (\(= 1/(2s_1) + 1/(2s_2)\)).

In the case, \(r < 2 < q\), we have

\[
1/\tilde{q} + 1/\tilde{r} = 1/q + 1 - 1/r > 1/q + 1 - (1/2 + 1/(d-1) + \varepsilon - 1/q) > 1/2,
\]

provided \(q < 2(d-1), \varepsilon < 2/q - 1/(d-1)\).

In the case, \(q < 2 < r\), the roles of \(q, r\) are symmetrical and one recovers

\[
1/\tilde{q} + 1/\tilde{r} > 1/2, \text{ provided } r < 2(d-1), \varepsilon < 2/r - 1/(d-1).
\]

In the remaining case \(q \geq 2, r \geq 2\), we simply use that

\[
1/\tilde{q} + 1/\tilde{r} = 1/q + 1/r \in (1/2 + 1/(d-1) - \varepsilon, 1/2 + 1/(d-1) + \varepsilon)
\]

which implies in particular \(1/\tilde{q} + 1/\tilde{r} > 1/2\), if \(\varepsilon < 1/(d-1)\).

With this choice of \(s_1, s_2\), we have by Corollary 2.3

\[
\left(\sum_j \|P_j f_k\|_{L^q(R^d)}^{2s_1}\right)^{1/2s_1} = \left(\sum_j \|P_j f_k\|_{L^q(R^d)}\right)^{1/\tilde{q}} \leq C\|f_k\|_{L^\infty}
\]

\[
\left(\sum_j \|P_j g_n\|_{L^r(R^d)}^{2s_2}\right)^{1/2s_2} \leq \left(\sum_j \|P_j g_n\|_{L^r(R^d)}\right)^{1/\tilde{r}} \leq C\|g_n\|_{L^\infty}.
\]

Taking into account \(1/p_1 + 1/p_2 = 1/2\), we get

\[
2^{d-1/2} \|T_i^{+\text{toy}}(f_k, g_n)\|_{L^2(R^d)} \lesssim 2^{d/2 + d(1/q - 1)/p_1} \|f_k\|_{L^p} 2^{n(1/r - 1/p_2)} \|g_n\|_{L^r}.
\]

(6.2)

Since \(k \leq n + 3\), it is beneficial to finally choose \(p_1 = \infty, p_2 = 2\) if \(r < 2\), and \(p_1 = 2r/(r-2), p_2 = r\), if \(r \geq 2\), which gives the estimate

\[
\sup_i 2^{d-1/2} \|T_i^{+\text{toy}}(f_k, g_n)\|_{L^2(R^d)} \lesssim 2^{kd(\alpha(d, q, r))} \|f_k\|_{L^p} 2^{n(1/r - 1/r - \alpha(d, q, r))} \|g_n\|_{L^r}
\]

valid for all \(q, r\) in a neighborhood of the fixed point \(q_1, r_1\). A real bilinear interpolation argument (see for example Exercise 5, p. 76, [2]) yields

\[
\sum_i 2^{d-1/2} \|T_i^{+\text{toy}}(f_k, g_n)\|_{L^2(R^d)} \lesssim 2^{kd(\alpha(d, q, r))} \|f_k\|_{L^p} 2^{n(1/q + 1/r - \alpha(d, q, r))} \|g_n\|_{L^r}
\]

for all \(q, r\) in the same neighborhood of \(q_1, r_1\), which is (5.5).

6.2. Proof of (5.3).

As pointed out earlier, the proof of (5.3) goes very similarly to (5.5). Starting with (5.6), we have

\[
\|T_i^{+}(f_k, g_n)\|_{L^2} \leq \sum_{h=0}^{\infty} \sum_j \bar{Q}_{j, l, m, h}(P_j^{l+m, \alpha}Q_{j; l, m, h}f_k) \|P_j^{l+m, \alpha}g_n\|_{L^2}
\]
Clearly, choosing 1 for every (2 supports in the first sum, a factor of \(q, r\) the last two sums are estimated as in Section 6.1, under the appropriate assumption.

Entering this estimate in the chain of inequalities above yields (2.1), (2.2)), we conclude that for every integer \(N\) and in particular, note that by the presence of the cutoff \(\phi \) given by

where

Next, due to the Fourier support properties of the multiplier \(\hat{Q}_{j,l,m,h}\) and Hölder's

and in particular, note that by the presence of the cutoff \(\phi(2^{-h}(\xi/|\xi| - \theta_j^{t+m}))\), we have \(|\xi/|\xi| - \theta_j^{t+m}| \sim 2^{-h}\), thus forcing the argument of the first term

Thus, by Lemma 2.2 (more specifically (2.7)) and Lemma 2.1 (more specifically (2.1), (2.2)), we conclude that for every integer \(N\), there exists \(C_{d,N}\), so that

Entering this estimate in the chain of inequalities above yields

The last two sums are estimated as in Section 6.1, under the appropriate assumptions on \(q, r\). Note however, that we get, due to the increased overlap in the Fourier supports in the first sum, a factor of \(2^{(m+h)(d-1)/(2s_1)}\). More precisely,

Clearly, choosing \(N > 4(d - 1) + 1\) allows us to sum the estimates in \(h\)

for every \((q, r)\) in a neighborhood of a point \((q_1, r_1)\) satisfying \(1/q_1 + 1/r_1 = 1/2 + 1/(d - 1)\). By identical real interpolation argument as in Section 6.1, we conclude
that in the same neighborhood,
\[ \sum_l 2^{l(d-1)(1/q+1/r-1/2)} \|T_l^+(f_k, g_n)\|_{L^2} \leq \]
\[ \leq C_d^{(\alpha+1/2-m)2k\alpha(d,q,r)} \|f_k\|_{L^{q,r}} 2^n d^{1/q+1/r-1/2-\alpha(d,q,r)} \|g_n\|_{L^{q,r}} \]
whence for \((q_1, r_1)\), one obtains
\[ \sum_l 2^l \|T_l^+(f_k, g_n)\|_{L^2} \leq C_d^{(\alpha+1/2-m)2k\alpha(d,q,r)} \|f_k\|_{L^{q,r}} 2^n d^{1/q+1/r-1/2-\alpha(d,q,r)} \|g_n\|_{L^{q,r}} \]
which is (5.3). As we have pointed out, that estimate by itself implies the statement of Theorem 4.1, since it allows one to sum in the \(\alpha\) and \(m\) variables. Regarding the claimed estimates for \(T^t\), we see from (5.7)
\[ \|T^{-,t_{0y}}(f, g)\|_{L^2(R^d)} = \sum_j \|P_j P_j f Z_j g\|_{L^2} \leq (\sum_j \|P_j f\|_{L^{p,1}_r}^{2s_1})^{1/2} (\sum_j \|Z_j g\|_{L^{r,s_2}}^{2s_2})^{1/2}, \]
for all \(p_1, p_2, s_1, s_2 : 1/p_1 + 1/p_2 = 1/2, 1/s_1 + 1/s_2 = 1\). But this is exactly equivalent\(^1\) to (6.1), whence the whole argument follows in an identical fashion. We get (4.8) and subsequently (4.7) for \(T^{-,t_{0y}}\) and by the arguments in Section 6.2 for \(T^t\) as well.

### 6.3. Estimates for \(T^t\) in the high-high case.

We briefly discuss the proof of (4.9) for the operator \(T^{+,t_{0y}}\). We have
\[ \|S_k[T^{+,t_{0y}}(f_{\sim n}, g_n)]\|_{L^2} = \sum_j \|S_k P_j P_{j-n} P_j g_n\|_{L^2} \]
Hence, by the Bernstein inequality (note that the Fourier support of \(S_k P_j\) has volume \(\sim 2^{kd} 2^{-(d-1)}\)), with \(1/2 = 1/p + \delta, \delta < 1/(d-1)\),
\[ \sum_j \|S_k P_j P_{j-n} P_j g_n\|_{L^2} \leq C_d 2^{kd} 2^{-(d-1)} 2^{p-1} \sum_j \|S_k P_j P_{j-n} P_j g_n\|_{L^p} \leq \]
\[ \leq C_d 2^{kd} 2^{2d-\delta} (\sum_j \|P_{j-n} f_{\sim n}\|_{L^{q,s_1}}^{2s_1})^{1/2} (\sum_j \|P_j g_n\|_{L^{r,s_2}}^{2s_2})^{1/2} \]
for all \(q, r : 1/q + 1/r = 1/p = 1/2 - \delta\) and \(1/s_1 + 1/s_2 = 1\). At this point, having gained the factor \(2^{kd}\), the proof proceeds as before. Say in the case, \(q, r \geq 2\), we obtain, similar to (6.2),
\[ 2^{d-1)(1/q+1/r-1/2)} \|S_k[T^{+,t_{0y}}(f_{\sim n}, g_n)]\|_{L^2} \leq C_{d,2} 2^{kd} 2^{n(d-1)-\delta} \|f_{\sim n}\|_{L^q} \|g_n\|_{L^r} \]
for all \(q, r \geq 2\). A bilinear interpolation gives for every \(q, r \geq 2 : 1/q + 1/r = 1/2 + 1/(d-1)\),
\[ \sum_l 2^l \|S_k[T^{+,t_{0y}}(f_{\sim n}, g_n)]\|_{L^2} \leq C_{d,2} 2^{kd} 2^{n(d-1)-\delta} \|f_{\sim n}\|_{L^{q,r}} \|g_n\|_{L^{q,r}} \]
\(^1\)Except that \(Z_j\) restricts to an antipodal neighborhood of \(\text{supp} \hat{P}_j\) with the same size, which is of course treated in an identical manner
7. Reducing the proof of Theorem 1.1 to multi-linear estimates involving the singular bilinear operator $T$

We now turn our attention back to the Dirac equation (3.8). Recall that we have a change of variables $U = V + |\nabla|^{-1} \Omega^{\pm};\pm(U^\pm, U^\pm)$. Thus, according to (4.6), for any $s$,

$$
\|U\|_{\mathcal{X}^s} \leq \|V\|_{\mathcal{X}^s} + \||\nabla|^{-1}\Omega^{\pm};\pm(U^\pm, U^\pm)\|_{\mathcal{X}^s} \leq \|V\|_{\mathcal{X}^s} + \||\nabla|^{-1}T(U, U)\|_{\mathcal{X}^s} + + \||\nabla|^{-1}T(|\nabla|^{-1}U, |\nabla|U)\|_{\mathcal{X}^s} + \|T(|\nabla|^{-1}U, U)\|_{\mathcal{X}^s}.
$$

Note that here, we are using the letter $T$ to denote either $T^+$ or $T^-$ and $U$ to denote either $U^+$ or $U^-$. This is not surprising, given the (almost) identical estimates satisfied by the operators, see Theorem 4.1.

Next, to relate $\|U\|_{\mathcal{X}^s}$ and $\|V\|_{\mathcal{X}^s}$, we will need to show\(^m\)

$$
\|\nabla|^{-1}T(u, v)\|_{\mathcal{X}^s} \lesssim \|u\|_{\mathcal{X}^s}\|v\|_{\mathcal{X}^s} + \|u\|_{\mathcal{X}^s}\|v\|_{\mathcal{X}^s}, \quad (7.1)
$$

$$
\|T(|\nabla|^{-1}u, v)\|_{\mathcal{X}^s} \lesssim \|u\|_{\mathcal{X}^s}\|v\|_{\mathcal{X}^s} + \|u\|_{\mathcal{X}^s}\|v\|_{\mathcal{X}^s}, \quad (7.2)
$$

$$
\||\nabla|^{-1}T(|\nabla|^{-1}u, |\nabla|v)\|_{\mathcal{X}^s} \lesssim \|u\|_{\mathcal{X}^s}\|v\|_{\mathcal{X}^s} + \|u\|_{\mathcal{X}^s}\|v\|_{\mathcal{X}^s}. \quad (7.3)
$$

for any two functions $u, v \in \mathcal{X}$. Then, (7.1), (7.2) and (7.3), imply

$$
\|U\|_{\mathcal{X}^s} \leq \|V\|_{\mathcal{X}^s} + C(\|U\|_{\mathcal{X}^s} + \|V\|_{\mathcal{X}^s})(\|U\|_{\mathcal{X}^s} + \|V\|_{\mathcal{X}^s}) \quad (7.4)
$$

We now continue with an estimate for $V$, which satisfies an equation in terms of $U$. More specifically, $V$ solves (3.8) and by the Strichartz estimates for the Dirac equation (2.5) with $\alpha = s - 1$, we get

$$
\|V\|_{\mathcal{X}^s} \lesssim \|V(0)\|_{\dot{B}^{2s+1}_{2,1}} + + \sum_k 2^{k(d/2+s-2)}\|S_k[\Omega^{\pm};\pm(|\nabla|^{-1}P[U^\pm \cdot \nabla U^\pm], U^\pm)]\|_{L^1L^2} + + \sum_k 2^{k(d/2+s-2)}\|S_k[\Omega^{\pm};\pm(U^\pm, |\nabla|^{-1}P[U^\pm \cdot \nabla U^\pm])]\|_{L^1L^2} + + \sum_k 2^{k(d/2+s-2)}\|S_k[\Omega^{\pm};\pm(|\nabla|^{-1}P[F(U^\pm)], U^\pm)]\|_{L^1L^2} + + \sum_k 2^{k(d/2+s-2)}\|S_k[\Omega^{\pm};\pm(U^\pm, |\nabla|^{-1}P[F(U^\pm)])]\|_{L^1L^2}
$$

To treat $\|V(0)\|_{\dot{B}^{2s+1}_{2,1}}$, recall that $V(0) = U(0) - |\nabla|^{-1}\Omega^{\pm};\pm(U^\pm, U^\pm)|_{t=0}$ and

\(^m\)Here, we are essentially using the symmetry of the operator $T$. While this is clearly the case for $T^+$, it is technically not so for $T^-$. However, one should note that the estimates in Theorem 4.1 are symmetric in $f$ and $g$ and this is the symmetry that is really used.
hence by (4.6),
\[ \|V(0)\|_{L^2} \leq \|U(0)\|_{L^2} + \|\nabla^{-1} \Omega^{\pm, \pm}(U^\pm, U^\pm)\|_{L^2} \lesssim \]
\[ \lesssim \|(f, g)\|_{L^2} \times \|\nabla^{-1} T(U^\pm, U^\pm)\|_{X^s} \]
\[ \lesssim \|(f, g)\|_{L^2} \times \|\nabla^{-1} T(U^\pm, U^\pm)\|_{X^s} + \|T(\nabla^{-1} U^\pm, U^\pm)\|_{X^s} \lesssim \]
where the last estimate is a consequence of (7.1), (7.2), (7.3). For the cubic terms, we use the representation formula (4.6) to estimate
\[ \sum_k 2^{k(d/2+s-1)} \|S_k[\Omega^{\pm, \pm}(U^\pm, U^\pm)]\|_{L^1 L^2} \lesssim \]
\[ \lesssim \sum_k 2^{k(d/2+s-1)} \|S_k[T(\nabla^{-1} \mathcal{P}[U^\pm, \nabla U^\pm])]\|_{L^1 L^2} + \]
\[ + \sum_k 2^{k(d/2+s-1)} \|S_k[T(\nabla^{-2} \mathcal{P}[U^\pm, \nabla U^\pm])]\|_{L^1 L^2} + \]
\[ \sum_k 2^{k(d/2+s-1)} \|S_k[T(\nabla^{-2} \mathcal{P}[U^\pm, \nabla U^\pm])]\|_{L^1 L^2}. \]
For the term \( \sum_k 2^{k(d/2+s-1)} \|S_k[\Omega^{\pm, \pm}(U^\pm, \nabla^{-1} \mathcal{P}[U^\pm, \nabla U^\pm])]\|_{L^1 L^2} \), we estimate in the same way through the representation formula (4.6). We get the same terms as in the previous one, except for
\[ \sum_k 2^{k(d/2+s-1)} \|S_k[T(\nabla^{-1} \mathcal{P}[U^\pm, \nabla U^\pm])]\|_{L^1 L^2}. \]
Hence, in addition to (7.1), (7.2), (7.3), we will also need to show
\[ \sum_k 2^{k(d/2+s-1)} \|S_k[T(\nabla^{-1} \mathcal{P}[U \cdot \nabla v, w])]\|_{L^1 L^2} \lesssim \]
\[ \lesssim \sum_k 2^{k(d/2+s-1)} \|S_k[T(\nabla^{-2} \mathcal{P}[U \cdot \nabla v, |v| w])]\|_{L^1 L^2} \lesssim \]
\[ \lesssim \sum_k 2^{k(d/2+s-1)} \|S_k[T(\nabla^{-2} \mathcal{P}[U \cdot \nabla v, w])]\|_{L^1 L^2} \lesssim \]
The last estimate that we shall need is more special\(^a\) in the sense that we need to
\(^a\)This is where we have to use the smallness of the initial data in the spaces \( \|f\|_{\dot{B}^{d/2+\alpha}_{d/2}} + \|g\|_{\dot{B}^{d/2+\alpha}_{d/2}} \) for \( \alpha \in [-1/2, 1/2] \)
transfer 1/2 derivative from the \( w \) term to either the \( u \) term or the \( v \) term,

\[
\sum_k 2^{k(\frac{d}{2}+s-2)} \| S_k [T(\mathcal{P}[u \cdot \nabla v], |\nabla|^{-1} w)] \|_{L^1 L^2} \lesssim \begin{cases} \\
\| u \|_X v \|_{X^0} \| w \|_{X^s} + \\
\| u \|_X v \|_{X^0} \| w \|_{X^0} + \\
\| u \|_X v \|_{X^s} w \|_{X^0} + \\
\| u \|_X v \|_{X^s} w \|_{X^s-1/2} + \\
\| u \|_X v \|_{X^s} w \|_{X^s-1/2}.
\end{cases}
(7.8)
\]

Finally, to handle the quartic term, we need to prove

\[
\sum_k 2^{k(\frac{d}{2}+s-2)} \| S_k [\Omega(|\nabla|^{-1} \mathcal{P}[uvw], z)] \|_{L^1 L^2} \lesssim \begin{cases} \\
\| u \|_X v \|_{X^0} \| w \|_{X^s} + \\
\| u \|_X v \|_{X^0} \| w \|_{X^0} + \\
\| u \|_X v \|_{X^s} w \|_{X^0} + \\
\| u \|_X v \|_{X^s} w \|_{X^s-1/2} + \\
\| u \|_X v \|_{X^s} w \|_{X^s-1/2}.
\end{cases}
(7.9)
\]

\[
\sum_k 2^{k(\frac{d}{2}+s-2)} \| S_k [\Omega(z, |\nabla|^{-1} \mathcal{P}[uvw])] \|_{L^1 L^2} \lesssim \begin{cases} \\
\| u \|_X v \|_{X^0} \| w \|_{X^s} + \\
\| u \|_X v \|_{X^0} \| w \|_{X^0} + \\
\| u \|_X v \|_{X^s} w \|_{X^0} + \\
\| u \|_X v \|_{X^s} w \|_{X^s-1/2} + \\
\| u \|_X v \|_{X^s} w \|_{X^s-1/2}.
\end{cases}
(7.10)
\]

Let us now prove Theorem 1.1, based on (7.4)-(7.10). Indeed, we have

\[
\|u\|_{X^s} - C\|u\|_{X^0} \|v\|_{X^0} \|v\|_{X^s} \|w\|_{X^0} \leq \|v\|_{X^s} \leq \|f\|_{B^{d/2+\frac{3}{2}}_{2,1}} + \|g\|_{B^{d/2+\frac{3}{2}}_{2,1}} \lesssim \|f\|_{B^{d/2+\frac{3}{2}}_{2,1}} + \|u\|_{X^0} \|v\|_{X^s} \|w\|_{X^0} \|v\|_{X^0} + \|u\|_{X^0} \|v\|_{X^s} \|w\|_{X^0} + \|u\|_{X^0} \|v\|_{X^s} \|w\|_{X^s-1/2} + \|u\|_{X^0} \|v\|_{X^s} \|w\|_{X^s-1/2}.
\]

Thus, for initial data \( f, g : sup_{\alpha \in [-1/2, 1/2]} (\|f\|_{B^{d/2+\frac{3}{2}}_{2,1}} + \|g\|_{B^{d/2+\frac{3}{2}}_{2,1}}) \ll 1 \), there exists a constant \( C_d \), so that \( \|U\|_{X^s} \leq C \|f\|_{B^{d/2+\frac{3}{2}}_{2,1}} + \|g\|_{B^{d/2+\frac{3}{2}}_{2,1}} \). This is the global regularity result.

Moreover, if one deals with two different solutions \( U^1, U^2 \), we subtract the corresponding equations (3.8) to conclude that the difference \( V^1 - V^2 \) satisfy a Dirac equation with a right-hand side, which is a linear combination of cubic terms exactly like (3.8), where one of the entries in each term is either \( u^1 - u^2 \) or \( V^1 - V^2 \) and the other entries are in the form \( w^j, V^j, j = 1, 2 \). Proceeding in the same fashion, by using (7.4), and then by (7.5), (7.6), (7.7), (7.8)

\[
\|U^1 - U^2\|_{X^s} \lesssim \|f^1 - f^2, g^1 - g^2\|_{B^{d/2+\frac{3}{2}}_{2,1}} + \|U^1 - U^2\|_{X^s} \|w^0 + \|U^2\|_{X^s} + \|U^1\|_{X^{1/2}}, x^{1/2} + \|U^2\|_{X^{1/2}}, x^{1/2})^2.
\]

Again, under the assumption \( sup_{\alpha \in [-1/2, 1/2]} (\|f\|_{B^{d/2+\frac{3}{2}}_{2,1}} + \|g\|_{B^{d/2+\frac{3}{2}}_{2,1}}) \ll 1 \), we obtain by continuity argument

\[
\|U^1 - U^2\|_{X^s} \leq C_d \|f^1 - f^2, g^1 - g^2\|_{B^{d/2+\frac{3}{2}}_{2,1}} \times B^{d/2+\frac{3}{2}}_{2,1},
\]

as announced in Theorem 1.1.
8. Proof of the tri-linear and quatri-linear estimates: (7.5), (7.6), (7.7), (7.8), (7.9), (7.10)

In this section, we stick to the case $d = 5$, which is the hardest case to deal with anyways. We split our considerations in cases, according to the following standard decomposition

$$S_k(fg) = S_k(f S \sim k g) + S_k(S \sim k f S \sim k g) + S_k(S \sim k f S \prec k g).$$

We refer to these as low-high, high-low and high-high interactions. Note that in addition to this decomposition, we can further write

$$S_k(f S \sim k g) = S_k(T(S \geq k + 3 f S \sim k g)),$$

since $S_k(T(S \geq k + 3 f S \sim k g)) = 0$ by Fourier support considerations. For the high-high interactions, we also have

$$S_k[T(S \prec k f, S \prec k g)] = \sum_{m \geq k+3} S_k[T(S \prec m f, S \prec m g)] + \sum_{m \geq k+3} S_k[T(f \sim m, g \sim m)].$$

We consider these cases separately.

8.1. Low-high interactions for (7.5), (7.6), (7.7), (7.8)

In this case, we consider the case when the frequency of the null form entry $P[u \cdot \nabla v]$ is majorized by the frequency of the $w$ entry. Out of the four estimates involved, clearly the most difficult term to is the one where the derivative falls on a the high-frequency term, i.e. (7.6) and (7.7). They are similar, so we consider (7.7). We need to show

$$\sum_k 2^{(3/2 + s) k} \|S_k[T(|\nabla|^{-2} S \leq k + 3 P[u \nabla v], S \sim k w)]\|_{L^1 L^2} \lesssim \|u\|_{X^0} \|v\|_{X^0} \|v\|_{X^s} \quad (8.1)$$

The left hand side of (8.1) is estimated by Theorem 4.1 and more precisely by (4.8) with $r = 2, q = 4$. Note that these exponents are on the sharp line and we need to apply (4.8).

$$\sum_k 2^{(3/2 + s) k} \|S_k[T(|\nabla|^{-2} S \leq k + 3 P[u \nabla v], S \sim k w)]\|_{L^1 L^2} \lesssim$$

$$\lesssim \sum_k \sum_{m \prec k+3} 2^{-3m/4} 2^{(3/2 + s) k} \|w \sim k\|_{L^\infty L^2} \|S_m[u \nabla v]\|_{L^1 L^{4,2}} \lesssim$$

$$\lesssim \|w\|_{X^s} \sum_m 2^{m/4} \|S_m[u v]\|_{L^1 L^{4,2}}.$$
Global regularity for Yang-Mills fields in $\mathbb{R}^{1+5}$

where we have used that $u \nabla v = \text{div}(uv)$. Thus, to finish off the proof of (8.1), it will suffice to check
\[
\sum_m 2^{m/4} \| S_m[uv] \|_{L^1 Y^2} \lesssim \| u \|_{X^0} \| v \|_{X^0}.
\] (8.2)

We verify (8.2) by looking at different type of interactions. For low-high interactions between $u$ and $v$ (the high-low case is symmetric), we have
\[
\sum_m 2^{m/4} \| S_m[u \sim_m v < m+3] \|_{L^1 Y^2} \lesssim \sum_m 2^{m/4} \| u \sim_m \|_{L^2 L^{20/3.2}} \| v < m+3 \|_{L^2 L^{10}} \lesssim \sum_m 2^{m/4} \| u \|_{X^0} \| v \|_{X^0}.
\]

This shows (8.2) in both the high-low and the low-high cases. In the high-high case, we have
\[
\sum_m 2^{m/4} \sum_{l > m+2} \| S_m[l_{uv}] \|_{L^1 Y^2} \lesssim \sum_l 2^{l/4} \| u \|_{L^2 L^{5.2}} \| v \|_{L^2 L^8} \lesssim \sum_l 2^{l/8} \| u \|_{X^0} \| v \|_{X^0}.
\]

Next, for the high-low case, we will consider the case of (7.8) separately, since it is somewhat different.

8.2. High-low interactions for (7.5), (7.6), (7.7)

In this case, we are considering the situation where the null form is in high-frequency mode and the $w$ entry is in lower mode. Clearly, the most unfavorable distribution of the derivatives (and hence the most difficult term to control) occurs in (7.5) and (7.7), but they are again similar. We need to show
\[
\sum_k 2^{(1/2+s)k} \| T(|\nabla|^{-1} S_{\sim_k} [w \cdot \nabla v], w_{< k+3} \|_{L^1 L^2} \lesssim (\| u \|_{X^0} \| v \|_{X^0} + \| u \|_{X^0} \| v \|_{X^0}) \| w \|_{X^0}
\]

First, by $\text{div}(u) = 0$, we have $u \cdot \nabla v = \text{div}(u \cdot v)$ and hence by Theorem 4.1 with $r = 2, q = 4$, we get
\[
\sum_k 2^{(1/2+s)k} \| S_k[T(S_{\sim_k} |\nabla|^{-1} \text{div}[u \cdot v], w_{< k+3} \|_{L^1 L^2} \lesssim \sum_k 2^{(1/2+s)k} \| S_{\sim_k} [uv] \|_{L^2 L^2} \| w_{< k+3} \|_{L^2 L^{5.2}} \lesssim \sup_k 2^{(s+1)k} \| S_{\sim_k} [uv] \|_{L^2 L^2} \| w_{< k+3} \|_{L^2 L^{5.2}} \lesssim \| u \|_{L^2 L^{10}} \| \nabla^{s+1} v \|_{L^\infty L^{5/2}} + \| \nabla^{s+1} u \|_{L^\infty L^{5/2}} \| v \|_{L^2 L^{10}} \| w \|_{X^0} \lesssim (\| u \|_{X^0} \| v \|_{X^0} + \| u \|_{X^0} \| v \|_{X^0}) \| w \|_{X^0}
\]
8.3. High-low interactions for (7.8)

We need to estimate

\[ \sum_k 2^{(1/2+s)k} \| T(S_{\sim k} [u \cdot \nabla v], |\nabla|^{-1} w_{< k+3}) \|_{L^1 L^2}. \]

By Theorem 4.1 with \( r = 2, q = 4 \),

\[ \sum_k 2^{(1/2+s)k} \| T(S_{\sim k} [u \cdot \nabla v], |\nabla|^{-1} w_{< k+3}) \|_{L^1 L^2} \lesssim \]

\[ \lesssim \sum_k 2^{(1/2+s)k} \| u \cdot \nabla v \|_{L^2 L^2} \sum_{l< k+3} 2^{l/4} \| w_l \|_{L^2 L^4}. \]

Clearly \( \sum_{l< k+3} 2^{l/4} \| w_l \|_{L^2 L^4} \lesssim \| w \|_{X^{-1/2}} \), while since

\[ [u \cdot \nabla v]_{\sim k} = S_{\sim k} [u_{< k+3} \cdot \nabla v_{< k} + u_{\sim k} \cdot \nabla v_{< k+3} + \sum_{l \geq k+3} u_l \cdot \nabla v_{\sim l}] \]

we have

\[ \sum_k 2^{(1/2+s)k} \| u \cdot \nabla v \|_{L^2 L^2} \lesssim \sup_k \| u_{< k+3} \|_{L^2 L^\infty} \sum_k 2^{(3/2+s)k} \| v_{\sim k} \|_{L^\infty L^2} + \]

\[ + \sum_k 2^{(3/2+s)k} \| u_{\sim k} \|_{L^\infty L^2} \sup_l \| u_l \|_{L^2 L^\infty} + \sup_l \| u_l \|_{L^2 L^\infty} \sum_l 2^{(3/2+s)l} \| u_{\sim l} \|_{L^\infty L^2} \lesssim \]

\[ \lesssim \| u \|_{X^{1/2}} \| v \|_{X^2} + \| u \|_{X^2} \| v \|_{X^{1/2}}, \]

whence (7.8) follows.

8.4. High-high interactions for (7.5), (7.6), (7.7), (7.8)

In this case, we have that the worst term occurs where we have the maximum number of negative derivatives falling onto the small frequency terms, i.e. in (7.5) and (7.6). We consider (7.5), since (7.6) is similar. It will suffice to establish

\[ \sum_m \sum_{k \geq m+3} 2^{m(1/2+s)} \| S_m T([|\nabla|^{-1} S_{\sim k} P[u \nabla v], S_k w]) \|_{L^1 L^2} \lesssim \| u \|_{X^1} \| v \|_{X^1} \| w \|_{X^1}. \]

For (8.3), we have by Theorem 4.1 with \( r = 2 \) and \( q = 4 \),

\[ \sum_m \sum_{k \geq m+3} 2^{m(1/2+s)} \| S_m T([|\nabla|^{-1} S_{\sim k} P[u \nabla v], S_k w]) \|_{L^1 L^2} \lesssim \]

\[ \lesssim \sum_k 2^{k(3/4+s)} \| w_k \|_{L^2 L^{5/2}} \| S_{\sim k} [u \nabla v] \|_{L^2 L^2} \lesssim \| u \|_{X^2} \| u \nabla v \|_{L^2 L^2} \]

\[ \lesssim \| u \|_{X^2} \| u \|_{L^2 L^0} \| \nabla v \|_{L^\infty L^{5/2}} \lesssim \| u \|_{X^2} \| v \|_{X^0} \| w \|_{X^0}. \]
8.5. Proof of quatri-linear estimates (7.9) and (7.10)

Denoting \( F = F(u,v,w) = \mathcal{P}[uvw] \) and taking into account the representation formula (4.6), (7.9) and (7.10) will follow from

\[
\sum_k 2^{(1/2+s)k} ||S_k[T(|\nabla|^{-1}F,z)]||_{L^1 L^2} \lesssim A(u,v,w,z) \tag{8.4}
\]

\[
\sum_k 2^{(1/2+s)k} ||S_k[T(|\nabla|^{-2}F,|\nabla|z)]||_{L^1 L^2} \lesssim A(u,v,w,z) \tag{8.5}
\]

\[
\sum_k 2^{(3/2+s)k} ||S_k[T(|\nabla|^{-2}F,z)]||_{L^1 L^2} \lesssim A(u,v,w,z) \tag{8.6}
\]

\[
\sum_k 2^{(1/2+s)k} ||S_k[T(F,|\nabla|^{-1}z)]||_{L^1 L^2} \lesssim A(u,v,w,z) \tag{8.7}
\]

where

\[
A(u,v,w,z) = \begin{cases} 
||u||_{X^0} ||v||_{X^0} ||w||_{X^0} ||z||_{X^0} + \\
||u||_{X^0} ||v||_{X^0} ||w||_{X^0} ||z||_{X^0} + \\
||v||_{X^0} ||v||_{X^0} ||w||_{X^0} ||z||_{X^0} + \\
||u||_{X^0} ||v||_{X^0} ||w||_{X^0} ||z||_{X^0}.
\end{cases}
\]

Before we proceed with the proofs of (8.4), (8.5), (8.6), (8.7) we need an elementary lemma.

**Lemma 8.1.** Let \( \delta > 0 \) and \( 1 < p,q,r < \infty : 1/q + 1/r = 1/p + \delta \). Then, there exists \( C_{\delta,d} \) so that for all \( 0 \leq \delta_1, \delta_2 : \delta_1 + \delta_2 = \delta \),

\[
\sum_m ||S_m[uvw]||_{L^p} \leq C_{\delta} \sum_k 2^{kd\delta_1} ||u_k||_{L^q} \sum_n 2^{nd\delta_2} ||v_n||_{L^r}
\]

The proof is elementary, one just have to look at the various interactions that occur. If one iterates the lemma to a product of three functions, as is the case under consideration, we get for all \( \delta > 0 \) and \( 1 < p,q,r,s < \infty : 1/q + 1/r + 1/s = 1/p + \delta \),

\[
\sum_m ||S_m[uvw]||_{L^p} \leq C_{\delta} \sum_k 2^{kd\delta/3} ||u_k||_{L^q} \sum_n 2^{nd\delta/3} ||v_n||_{L^r} \sum_l 2^{ld\delta/3} ||w_l||_{L^s}
\tag{8.8}
\]

We turn now back to the proof of the quatri-linear estimates. In the low-high interaction case, the most difficult case occurs in (8.5) and (8.6). As before, they are similar, so we concentrate on (8.6). We estimate the left-hand side by Theorem 4.1 with \( r = 2, q = 4 \),

\[
\sum_k 2^{3/2+s} ||F_k||_{L^1 L^4,2} \lesssim ||z||_{L^2} \sum_l 2^{-3l/4} ||F_l||_{L^1 L^4,2} \lesssim ||z||_{X^0} \sum_l 2^{-3l/4} ||F_l||_{L^1 L^4,2} \lesssim ||z||_{X^0} \sum_l ||F_l||_{L^1 L^5/2} \lesssim ||u||_{X^0} ||v||_{X^0} ||w||_{X^0} ||z||_{X^0},
\]
where in the last step, we have used Lemma 8.1 with \( q = r = s = 5, p = 5/2, \delta = 1/5 \), which yields

\[
\sum_l ||F_l||_{L^1 L^{5/2}} \lesssim \sum_k 2^{k/3} ||u_k||_{L^3 L^5} \sum_l 2^{l/3} ||v_l||_{L^3 L^5} \lesssim \|

In the high-low interaction case, the hardest terms to control is in (8.7). We estimate the left-hand side of (8.7) by Theorem 4.1 with \( r = 2, q = 4, \delta = 1 \):

\[
\sum_k 2^{(4+s)k} ||S_k T(F_{j,k}, |\nabla|^{-1} z_{<k+3})||_{L^1 L^2} \lesssim \sum_k 2^{k/2} ||F_{j,k}||_{L^1 L^2} \sum_l 2^{l/4} ||z_l||_{L^\infty L^4}
\]

Since \( \sum_l 2^{l/4} ||z_l||_{L^\infty L^4} \lesssim ||z||_{X^0} \), it remains to show

\[
\sum_k 2^{(4+s)k} ||F_{j,k}||_{L^1 L^2} \lesssim ||u||_{X^0} ||v||_{X^0} ||\lambda^*|| ||u||_{X^0} ||v||_{X^0} ||\lambda^*|| ||z||_{X^0} + ||u||_{X^0} ||v||_{X^0} ||\lambda^*|| ||z||_{X^0} + ||u||_{X^0} ||v||_{X^0} ||\lambda^*|| ||z||_{X^0}.
\]

as desired.

Finally, in the high-high interaction situation, (8.4) and (8.5) are the hardest. For (8.4), we estimate by the high-high estimate (4.9) in Theorem 4.1 with \( r = 2, q = 4, \delta = 1/8, \)

\[
\sum_k 2^{(4+s)k} \sum_{l>k+3} ||T(|\nabla|^{-1} F_{j,l}, z_l)||_{L^1 L^2} \lesssim \sum_k 2^{(9/8+s)k} \sum_{l>k+3} 2^{-3l/8} ||F_{j,l}||_{L^2 L^2} ||z_l||_{L^1 L^2} \lesssim \sum_k 2^{(3/4+s)k} ||z_l||_{L^2 L^2} \sup_l ||F_{j,l}||_{L^2 L^2} \lesssim \|

9. Proof of the bilinear estimates (7.1), (7.2), (7.3)

First, in \( d \geq 3 \), the set of wave admissible pairs is the closed segment in the \((1/q, 1/r)\) plane, with endpoints \((0, 1/2)\) and \((1/2, (d-3)/2(d-1))\). Therefore, by the Gagliardo-Nirenberg inequality,

\[
||\phi||_{X^*} \lesssim \sum_k 2^{(3/4+s)k} ||f_k||_{L^2 L^2(\mathbb{R}^d)} + 2^{(3/2+s)k} ||f_k||_{L^\infty L^2(\mathbb{R}^d)}.
\]

We will consider the two extreme cases separately.

\(^a\)Here the difference with (8.8) is that we have to add \(1/2 + s\) derivative to either \(u, v, w\).
9.1. \( L^\infty L^2 \) estimates

We again divide into low-high or high-low interactions or high-high interactions.

9.1.1. Low-high interactions and high-low interactions

For the high-low case, the least favorable case is (7.1). We have by Theorem 4.1 with \( r = 2, q = 4 \)
\[
\sum_k 2^{(1+s)k} \| T(u_{r\leq k} + v_{r\sim k}) \|_{L^\infty L^2} \lesssim \sum_k 2^{(1+s)k} \| v_{r\sim k} \|_{L^\infty L^2} \sum_{l < k+3} 2^{5l/4} \| u_l \|_{L^\infty L^4} \\
\lesssim \sum_k 2^{(3/2+s)k} \| v_{r\sim k} \|_{L^\infty L^2} \| u_{r\sim k} \|_{L^\infty L^4} \lesssim \| u \|_{X^s} \| v \|_{X^0}. 
\]

For the low-high case, (7.2) and (7.3) are the hardest to control. For both, it will suffice to show
\[
\sum_k 2^{(3/2+s)k} \| T(\nabla^{-1} S_{r\leq k} u, S_{r\sim k} v) \|_{L^\infty L^2} \lesssim \| u \|_{X_t} \| v \|_{X^s}. 
\]

We estimate via (4.8) with \( q = 4 \) and \( r = 2 \). The left-hand side of (9.1) is bounded by
\[
\sum_k 2^{(3/2+s)k} \sum_{l < k+3} 2^{l/4} \| u_l \|_{L^\infty L^4} \| v_{r\sim k} \|_{L^\infty L^2} \lesssim \| u \|_{X_t} \| v \|_{X^s}. 
\]

9.1.2. High-High interactions

In this case, the difficult terms are (7.1) and (7.2). We will need to show
\[
\sum_m \sum_{k > m+3} 2^{(1/2+s)m} \| S_m [T(S_{r\sim k} u, S_k v)] \|_{L^\infty L^2} \lesssim \| u \|_{X_t} \| v \|_{X^s}. 
\]

We estimate by (4.9) with \( r = 2, q = 4, \delta = 1/8 \).
\[
\sum_m 2^{(1/2+s)m} \sum_k \| S_m [T(S_{r\sim k} u, S_k v)] \|_{L^\infty L^2} \lesssim \sum_m 2^{(9/8+s)m} \sum_{k > m+3} 2^{5k/8} \| S_m \|_{L^\infty L^4} \| v_k \|_{L^\infty L^2} \lesssim \sum_k 2^{(3/2+s)k} \| v_k \|_{L^\infty L^2} \| u_{r\sim k} \|_{L^\infty L^4} \| u \|_{X_t} \| v \|_{X^s}. 
\]

where we have used that \( 9/8 + s > 0 \) to sum in \( m \).

9.2. \( L^4_t L^4_x \) estimates

We need a lemma, which is the needed variant of the estimates in Theorem 4.1, in the case where the target is \( L^4_t \). We intentionally state the theorem in somewhat less general form (and for \( d = 5 \)), which will be used in the application below.
Lemma 9.1. Let $m,k$ be integers, $m < k + 3$. Then for every $q,r \geq 2 : 5/8 \leq 1/q + 1/r$

\[ \|T(f_m, g_k)\|_{L^{4/2}(\mathbb{R}^5)} \leq C2^{5m/q}2^{5k(1/r-1/4)}\|f_m\|_{L^{4/2}(\mathbb{R}^5)}\|g_k\|_{L^{r/2}(\mathbb{R}^5)} \] (9.3)

where $T$ is either $T^+$ or $T^-$. The proof follows the strategy of Theorem 4.1, with the additional complication that with the target space $L^{4/2}$, we cannot fully exploit the angular separation of variables, but in fact only (2.9). We will consider the toy versions of the operators associated to $T^+$ and $T^-$ (from (5.4) and (5.7) respectively), since the full operators are analyzed in the same fashion with a few extra sums.

Proof. We consider only $T = T^{+,\text{toy}}$, since the proof for $T^{-,\text{toy}}$ is almost identical. Indeed, we only use the disjointness of the Fourier supports of $\{P_j\}$ for $T^+$ and the disjointness of the Fourier supports $\{P_j\}$ and $\{Z_j\}$ for $T^-$. It suffices to show

\[ \sum_{l \geq 0} 2^l \|T^{+,\text{toy}}_l(f_m, g_k)\|_{L^{4/2}(\mathbb{R}^5)} \leq C2^{5m/q}2^{5k(1/r-1/4)}\|f_m\|_{L^{r/2}(\mathbb{R}^5)} \] (9.4)

whenever $q,r \geq 2 : 1/q + 1/r \geq 5/8$.

By (2.9) and Hölder’s inequality

\[ \|T^{+,\text{toy}}_l(f_m, g_k)\|_{L^{4/2}} = \| \sum_j P_j f_m P_j g_k \|_{L^4} \lesssim \left( \sum_j \|P_j f_m\|_{L^{4/3}}^4 \|P_j g_k\|_{L^{4/3}}^4 \right)^{3/4} \]

We then use the Bernstein inequality

\[ \|P_j f_m\|_{L^\infty} \lesssim 2^{4(m-l)+m}/q \|P_j f_m\|_{L^q} \]
\[ \|P_j g_k\|_{L^{4/2}} \lesssim 2^{4(k-l)+k(1/r-1/4)} \|P_j g_k\|_{L^{r/2}} \]

Thus,

\[ \|T^{+,\text{toy}}_l(f_m, g_k)\|_{L^{4/2}} \lesssim 2^{5m+5k(\frac{1}{4} - \frac{1}{2}) - 4l(\frac{1}{4} + \frac{1}{2})} \left( \sum_j \|P_j f_m\|_{L^q}^4 \|P_j g_k\|_{L^{r/2}}^4 \right)^{3/4} \]

In order to control the $j$ sums by a constant multiple of $\|f_m\|_{L^q}\|g_k\|_{L^{r/2}}$, we apply the Hölder’s inequality in $j$. Recall however, that the sum in $j$ is over points of the unit sphere $S^4$, approximately $2^{-l}$ distance away, so the numbers of the summands in $\sum_j$ is $\sim 2^{4l}$. We get by (2.8) of Corollary 2.3 (for $2 < q,r < \infty: 1/q + 1/r \leq 3/4$)

\[ \sum_j \|P_j f_m\|_{L^q}^4 \|P_j g_k\|_{L^{r/2}}^4 \leq \left( \sum_j \|P_j f_m\|_{L^q}^{q} \right)^{4/3} \left( \sum_j \|P_j g_k\|_{L^{r/2}}^{r} \right)^{4/(3r)} \times \]
\[ \times \left( \sum_j 1 \right)^{-\frac{1}{3}(1/q+1/r)} \lesssim 2^{4l(\frac{1}{4} + \frac{1}{2})}\|f_m\|_{L^q}^4\|g_k\|_{L^{r/2}}^{4/3}. \]

Putting this estimate together with the one before yields

\[ 2^{8(1/q+1/r)-4}\|T^{+,\text{toy}}_l(f_m, g_k)\|_{L^{4/2}} \lesssim 2^{5m/q}2^{5k(1/r-1/4)}\|f_m\|_{L^q}\|g_k\|_{L^{r/2}}. \] (9.5)
This estimate is valid for all $2 < q, r < \infty$ : $1/q + 1/r \leq 3/4$. In particular, it holds in an open neighborhood of any point $(1/q, 1/r)$ with $3/4 \geq 1/q + 1/r \geq 5/8$. Note that if $1/q + 1/r = 5/8$, we get a power of $2^l$ on the left-hand side of (9.5). Hence, by real bilinear interpolation, we obtain (9.4) and hence (9.3). Clearly the requirement $1/q + 1/r \leq 3/4$ is not needed and the estimates are actually quite good in that case. In fact, one picks up a power of at least $2^{2l}$ in (9.5).

Lemma 9.1 allows us to treat various cases occurring in (7.1).

9.2.1. Low-high interactions and high-low interactions
For these type of interactions, the worst terms are (7.2) and (7.3). Therefore, it will suffice to show

$$\sum_k 2^{(3/4+s)k} \|S_k T(\nabla^{-1} S_{\leq k+3} u, S_{\sim k} v)\|_{L^2 L^{4,2}}(\mathbb{R}^5) \lesssim \|u\|_{X^0} \|v\|_{X^s}$$  \hspace{1cm} (9.6)

Apply (9.3) from Lemma 9.1 with $r = 4$, $q = 8/3$. We obtain

$$\sum_k 2^{(3/4+s)k} \|S_k T(S_{\leq k+3} \nabla^{-1} u, S_{\sim k} v)\|_{L^2 L^{4,2}} \leq$$

$$\leq C \sum_k 2^{(3/4+s)k} \left( \sum_{l \leq k+3} 2^{7l/8} \|u_l\|_{L^{8/3}} \|v_{\sim k}\|_{L^2 L^{4,2}} \leq \|u\|_{X^0} \|v\|_{X^s} \right).$$

9.2.2. High-High interactions
As usual, we tackle only the cases with the least favorable distribution of the derivatives. Out of the three estimates, that occurs in (7.1). It will suffice to prove

$$\sum_m \sum_{k \geq m+3} 2^{(s-1/4)m} \|S_m T(u_{\sim k}, v_k)\|_{L^2 L^{4,2}} \lesssim \|u\|_{X^0} \|v\|_{X^s}.$$ 

We estimate by Sobolev embedding and Theorem 4.1 with $r = 2$, $q = 4$. We obtain

$$\sum_m \sum_{k \geq m+3} 2^{(s-1/4)m} \|S_m T(u_{\sim k}, v_k)\|_{L^2 L^{4,2}} \lesssim$$

$$\lesssim \sum_m 2^{(s+1)m} \sum_{k \geq m+3} \|T(u_{\sim k}, v_k)\|_{L^2 L^2} \lesssim \sum_k 2^{(s+9/4)k} \|u_{\sim k}\|_{L^2 L^{4,2}} \|v_k\|_{L^\infty L^2} \lesssim$$

$$\lesssim \left( \sum_k 2^{3(2s+1)k} \|v_k\|_{L^\infty L^2} \right) \sup_k 2^{3k/4} \|u\|_{L^2 L^{4,2}} \lesssim \|u\|_{X^0} \|v\|_{X^s}.$$

where we have used that $s + 1 > 0$ to sum in $m$.

**Acknowledgments:** The author wishes to thank Andrea Nahmod for the interesting heuristics regarding the normal form method, which are summarized in Section 3.1. It is also a pleasure to acknowledge several helpful conversations with Rodolfo Torres regarding bilinear homogeneous multipliers as in Lemma 2.4. The author acknowledges gratefully support from NSF-DMS 0701802 and NSF-DMS 0908802.
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