MATH 960: PROJECT V
DUE: MAY 11^{th}

(1) Let $K: X \to Y$ be compact and $x_n \to x$ weakly in X. Prove that $\lim_n \|Kx_n - Kx\| = 0$.

(2) Let X be a separable reflexive Banach space, Y is a Banach space and $K: X \to Y$ be a linear operator, so that whenever $x_n \to x$ weakly, then Kx_n converges strongly. Prove that $K: X \to Y$ is compact (and then it follows that $Kx_n \to Kx$ by the previous problem).

(3) Let $K: [0,1] \times [0,1] \to \mathscr{C}$ be so that $\int_0^1 \int_0^1 |K(s,t)|^2 ds dt < \infty$ and $\overline{K(s,t)} = K(t,s)$. Show that the corresponding Hilbert-Schmidt operator $T_K: L^2[0,1] \to L^2[0,1]$

$$T_K f(t) = \int_0^1 K(s,t) f(s) ds$$

is self-adjoint. In addition, prove that its eigenvalues and eigenvectors λ_j, x_j : $T_K x_j = \lambda_j x_j, \|x_j\| = 1$ satisfy

$$K(s,t) = \sum_j \lambda_j \overline{x_j(s)} x_j(t).$$

(4) Let $T: H \to H$ be self-adjoint operator on the Hilbert space H, $T \geq 0$ and T^2 is compact. Prove that T is compact as well.
Hint: Use that for $Q = T^2$, there is a complete orthonormal basis $\{x_n\}$ and eigenvalues $\lambda_n \geq 0$ (why?), so that

$$Qx = \sum_{n=1}^{\infty} \lambda_n \langle x, x_n \rangle x_n.$$

Then, show that $Tx = \lim_{N \to \infty} \sum_{n=1}^{N} \sqrt{\lambda_n} \langle x, x_n \rangle x_n$, where the limit is to be understood in the operator norm.

(5) (J .L. Lions lemma) Let $X \subset Y \subset Z$ are three Banach spaces, so that $i: X \to Y, i(x) = x$ is compact and $j: Y \to Z, j(y) = y$ is continuous. Prove that for every $\epsilon > 0$, there exists C_ϵ, so that for every $u \in X$,

$$\|u\|_Y \leq \epsilon \|u\|_X + C_\epsilon \|u\|_Z.$$

Hint: Argue by contradiction. That is, there exists $\epsilon_0 > 0$, so that for all n, there is $x_n \in X$, so that

$$\|x_n\|_Y \geq \epsilon_0 \|x_n\|_X + n\|x_n\|_Z.$$

Consider $y_n := \frac{x_n}{\|x_n\|_X}$.

(6) Prove that for every $\epsilon > 0$, there is C_ϵ, so that

$$\max_{x \in [0,1]} |u(x)| \leq \epsilon \max_{x \in [0,1]} |u'(x)| + C_\epsilon \|u\|_{L^1[0,1]}.$$

Hint: Use J.L. Lions lemma. You have to note in advance that

$$C^1[0,1] = \{f : [0,1] \to \mathscr{C} : f, f' \in C[0,1]\}; \|f\|_{C^1[0,1]} = \sup_{x\in[0,1]} |f'(x)| + \sup_{x\in[0,1]} |f(x)|.$$

is compactly embedded in $C[0,1]$ (by Arzela-Ascoli, why?)

(7) Let $T : H \to H$ is self-adjoint. Prove that $\lambda \in \rho(T)$ (i.e. $\lambda I - T$) is invertible) if and only if there exist[1] closed subspaces H_1, H_2, so that H_1, H_2 are T invariant, with $H = H_1 \oplus H_2$, so that

$$\sup_{\|x\|=1, x\in H_1} \langle Tx, x\rangle < \lambda < \inf_{\|y\|=1, y\in H_2} \langle Ty, y\rangle$$

Hint: For the sufficiency, use that $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$, where $T_j = T|_{H_j}$.

For the necessity, consider a continuous function,

$$f_1(x) = \begin{cases} 1 & x < \lambda - \epsilon \\ 0 & x > \lambda + \epsilon \end{cases}$$

and $f_2(x) = 1 - f_1(x)$. Note $f_1, f_2 \in C(\sigma(T)) : f_j^2(x) = f_j(x), x \in \sigma(T)$ for $0 < \epsilon << 1$. Take the projections $P_j = f_j(T)$ and

$$H_j := P_j[H].$$

Alternatively, for the necessity, we might show that $(T - \lambda I)$ is invertible on H_1 and on H_2 by using the result stating that $\sigma(M) \subset [\inf_{\|x\|=1} \langle Mx, x\rangle, \sup_{\|x\|=1} \langle Mx, x\rangle]$

[1]It is allowed that $H_1 = \emptyset$ or $H_2 = \emptyset$