MATH 960: PROJECT IV
DUE: APRIL 27^{th}

(1) Let $(\mathscr{A}, \|\cdot\|)$ be a Banach space, with a product operation $(x,y) \to x.y$ that satisfies all the standard distribubutive, associativity laws etc. In addition, assume that $|||x|||$

$$\|xy\| \leq M\|x\|\|y\|,$$

for all $x, y \in \mathscr{A}$ and for some $M > 1$. Assume that $\mathscr{A}$ has an unit element e : $\|e\| = 1$. Prove that $\mathscr{A}$ is a Banach algebra. In other words, show that there is an equivalent norm $|||\cdot|||$, so that

$$|||xy||| \leq |||x|||\,|||y|||$$

Hint: Try

$$|||x||| = \sup_{y\neq 0, y\in\mathscr{A}} \frac{\|xy\|}{\|y\|}.$$

Solution:
For $|||x|||$, we clearly have $|||ax||| = |a|\,|||x|||$ and $|||x||| = 0$ if and only if $x = 0$. Next, by the triangle inequality for $\|\cdot\|$,

$$|||x_1 + x_2||| = \sup_{y\neq 0, y\in\mathscr{A}} \frac{\|(x_1+x_2)y\|}{\|y\|} \leq \sup_{y\neq 0, y\in\mathscr{A}} \frac{\|x_1 y\|}{\|y\|} + \sup_{y\neq 0, y\in\mathscr{A}} \frac{\|x_2 y\|}{\|y\|} = |||x_1||| + |||x_2|||$$

Note that the definition of $|||\cdot|||$ can be interpreted in the form

$$|||x||| = \sup_{z\neq 0, z\in\mathscr{A}:\|z\|=1} \|xz\|.$$

So,

$$\begin{aligned} |||x_1 x_2||| &= \sup_{z\neq 0, z\in\mathscr{A}:\|z\|=1} \|x_1 x_2 z\| = \sup_{z\neq 0, z\in\mathscr{A}:\|z\|=1} \frac{\|x_1(x_2 z)\|}{\|x_2 z\|}\|x_2 z\| \leq \\ &\leq \sup_{z\neq 0, z\in\mathscr{A}:\|z\|=1} \frac{\|x_1(x_2 z)\|}{\|x_2 z\|} \sup_{z\neq 0, z\in\mathscr{A}:\|z\|=1} \|x_2 z\| \leq |||x_1|||\,|||x_2|||. \end{aligned}$$

Finally,

$$\|x\| = \frac{\|xe\|}{\|e\|} \leq |||x||| \leq M\|x\|,$$

so it is equivalent norm.

(2) Let $\mathscr{A}$ be an unital Banach algebra, not necessarily commutative. Let $M \in \mathscr{A}$ be an element, that satisfies $p(M) = 0$ for some polynomial p. Assume that this polynomial is of minimal degree, i.e. for any polynomial $q : deg(q) < deg(p)$, we have that $q(M) \neq 0$. Prove that

$$\sigma(M) = \{\lambda \in \mathscr{C} : p(\lambda) = 0\}.$$

Solution: By the spectral mapping theorem, $\sigma(p(M)) = p(\sigma(M))$, whence $p(\sigma(M)) = \sigma(p(M)) = \sigma(0) = \{0\}$.

$$\sigma(M) \subset \{\lambda \in \mathscr{C} : p(\lambda) = 0\}.$$

To show the equality, assume for a contradiction that that $\lambda_0 : p(\lambda_0) = 0$, but $\lambda_0 \notin \sigma(M)$. Then, there is a polynomial $q : deg(q) = deg(p) - 1$, so that

$$p(\lambda) = (\lambda - \lambda_0) q(\lambda).$$

Thus,

$$q(M)(M - \lambda_0 I) = p(M) = 0$$

Since $(M - \lambda_0 I)$ is invertible, apply $(M - \lambda_0 I)^{-1}$ to the previous identity. It follows that $q(M) = 0$, a contradiction with the minimality of p.

(3) Extend the previous result to analytic functions p. More precisely, let there be an analytic function $g \in H(\Omega)$, so that $g(M) = 0$, where Ω is an open set containing $\sigma(M)$. Consider the set

$$H_M = \{G \in H(\Omega) : G(M) = 0\},$$

and introduce an order relation $G_1 \preceq G_2$, if $G_2/G_1 \in H(\Omega)$. Let p be a minimal element with respect to this order in H_M. Prove that

$$\sigma(M) = \{\lambda \in \mathscr{C} : p(\lambda) = 0\}.$$

Solution:
Similar to the previous one, by the spectral mapping theorem,

$$\sigma(M) \subset \{\lambda \in \mathscr{C} : p(\lambda) = 0\}.$$

Assume for a contradiction that that $\lambda_0 : p(\lambda_0) = 0$, but $\lambda_0 \notin \sigma(M)$. Again, there is a $q \in H(\Omega)$, so that $p(\lambda) = (\lambda - \lambda_0) q(\lambda)$, namely

$$q(\lambda) = \begin{cases} \frac{p(\lambda) - p(\lambda_0)}{\lambda - \lambda_0} & \lambda \neq \lambda_0, \lambda \in \Omega \\ p'(\lambda_0) & \lambda = \lambda_0 \end{cases}$$

This is indeed an analytic function by Riemann removable singularity theorem. Again

$$q(M)(M - \lambda_0 I) = p(M) = 0$$

implying $q(M) = 0$, since $\lambda_0 \notin \sigma(M)$. But clearly $\frac{p}{q} = \lambda - \lambda_0 \in H(\Omega)$, whence $q \preceq p$, a contradiction with the minimality of p.

(4) Find the spectrum of the operator $T : l^p \to l^p$, $1 \leq p \leq \infty$

$$T(x_1, x_2, x_3, x_4, \ldots) = (-x_2, x_1, -x_4, x_3, \ldots).$$

Solution:
We can apply the previous result since $I + T^2 = 0$, whence

$$\sigma(T) = \{z : z^2 + 1 = 0\},$$

whence $z = \pm i$.

(5) Show that if in a Banach algebra A, we have $\|y\|_A = |\sigma(y)|$, for every $y \in A$, then A is commutative.
Hint: This is the reverse of one of the theorems that we have established in class (in the C^* context).
Follow the following steps:
- Show that for every $w \in A$ invertible, $\sigma(w^{-1}xw) = \sigma(x)$. This is true in general and does not use $\|x\|_A = |\sigma(x)|$.
- Conclude that $\|x\|_A = \|w^{-1}xw\|_A$.
- Use the previous step for $w = e^{\lambda x}$ for $\lambda \in C$ in conjunction with the Liouville's theorem ("every bounded entire function is a constant") to conclude
$$e^{\lambda x}y = ye^{\lambda x}.$$
- Compare the coefficients in the previous identity.

Solution:
We have for every $\lambda \in \mathscr{C}$,
$$w^{-1}xw - \lambda e = w^{-1}(x - \lambda e)w$$
Thus, $w^{-1}xw - \lambda e$ is invertible, if $x - \lambda e$ is invertible and vice versa. Thus $\rho(w^{-1}xw) = \rho(x)$, and hence $\sigma(w^{-1}xw) = \sigma(x)$. Thus,
$$\|x\| = |\sigma(x)| = |\sigma(w^{-1}xw)| = \|w^{-1}xw\|.$$
Hence the A valued entire function $\lambda \to e^{-\lambda y}xe^{\lambda y}$ satisfies
$$\|e^{-\lambda y}xe^{\lambda y}\| = \|x\|.$$
By Lioville's theorem, it is bounded - one could argue that for every element $l \in A^*$, $f(\lambda) = l(e^{-\lambda y}xe^{\lambda y})$ is a bounded entire function and hence constant. Thus, $f(\lambda) = f(0) = l(x)$. It follows that
$$l(e^{-\lambda y}xe^{\lambda y} - x) = 0.$$
By Hahn-Banach ($l \in A^*$ separate the points in A), $e^{-\lambda y}xe^{\lambda y} - x = 0$ or $e^{\lambda y}x = xe^{\lambda y}$. Taking a derivative at zero at the last identity reveals that $xy = yx$.

(6) Suppose that in a Banach algebra, $\|xy\| \le M\|yx\|$ for some constant M. Prove that A is commutative.
Solution:
The problem here is the same. Indeed, setting $y = e^{\lambda u}, x = e^{-\lambda u}v$, for arbitrary $u, v \in A$ and $\lambda \in \mathscr{C}$, yields
$$\|e^{-\lambda u}ve^{\lambda u}\| \le M\|v\|.$$
As in the previous problem, this implies $e^{\lambda u}v = ve^{\lambda u}$, whence $uv = vu$.

(7) Let $\{\mathscr{A}\}$ be a C^* algebra and $x = x^*$. Prove that at least one of the reals $\|x\|, -\|x\|$ belong to $\sigma(x)$.
Solution:
Since $x = x^*$, $\sigma(x) \subset \mathbf{R}$ and $|\sigma(x)| = \|x\|$. Now, there is a sequence $\lambda_n \in \sigma(x)$, so that $|\lambda_n| \to |\sigma(x)| = \|x\|$. Since λ_n are real, it follows that a subsequence goes to either $-\|x\|$ or $\|x\|$ (or both). Say $\lambda_{n_k} \to \|x\|$. Since $\sigma(x)$ is closed, it follows that $\|x\| = \lim_k \lambda_{n_k}$ also belongs to $\sigma(x)$. Similar for the other case, $\lambda_{n_k} \to -\|x\|$.

(8) For a sequence of weights, $(w_n)_{n=-\infty}^{\infty}$ satisfying $w_0 = 1$, $0 < w_{m+n} \le w_m w_n$, define the space of complex sequences,

$$A = \{(f_n)_n \, \|f\|_A = \sum_n |f_n| w_n\}$$

Show that the product operation

$$(f * g)_n = \sum_{k=-\infty}^{\infty} f_{n-k} g_k,$$

turns A into a *commutative Banach algebra*. For the sequence (w_n) via $w_n = 2^n, n \ge 0$ and $w_n = 1, n < 0$, identify the multiplicative linear functionals on A and compute the spectrum for each element of A.

Hint: It is good to think again

$$A = \{f : [0,1] \to \mathscr{C} : f = \sum_n f_n e^{2\pi i n x}, \|f\|_A = \sum_n |f_n| w_n < \infty\}$$

with point-wise multiplication for the functions. Your answer should contain the term Laurent series.

Solution:

We have that

$$(f * g)_n = \sum_{k=-\infty}^{\infty} f_{n-k} g_k = \sum_{l=-\infty}^{\infty} f_l g_{n-l} = (g * f)_n$$

The triangle inequality and the homogeneity of the norm are obvious. The product inequality goes as follows

$$\begin{aligned} \|f * g\|_A &= \sum_n |(f * g)_n| w_n \le \sum_n |\sum_{k=-\infty}^{\infty} f_{n-k} g_k| w_n \le \sum_n \sum_{k=-\infty}^{\infty} |f_{n-k}||g_k| w_n \\ &\le \sum_n \sum_{k=-\infty}^{\infty} |f_{n-k}||g_k| w_{n-k} w_k = \sum_l |f_l| w_l \sum_k |g_k| w_k = \|f\|_A \|g\|_A. \end{aligned}$$

Regarding the multiplicative linear functionals, introduce

$$\lambda = p(e^{2\pi i x}),$$

so that $\lambda^n = p(e^{2\pi i n x})$. Thus, λ needs to satisfy

$$|\lambda|^n = |p(e^{2\pi i n x})| \le \|e^{2\pi i n x}\|_A = w_n$$

Thus, $|\lambda| \le \inf_n |w_n|^{1/n}$ for $n > 0$ and $|\lambda| \ge \sup_{n>0} \frac{1}{w_{-n}^{1/n}}$. For the example that we are considering, this amount to the following necessary conditions for λ:

$$1 \le |\lambda| \le 2.$$

On the other hand, these conditions are also sufficient, since the expression

$$\sum_{n=-\infty}^{\infty} f_n \lambda^n$$

converges for $(f_n) \in A$, $\lambda : 1 \le |\lambda| \le 2$. This is the Laurent series with coefficients provided by f_n. Thus, all m.l.f. on A are given by

$$p_\lambda(f) = \sum_{n=-\infty}^{\infty} f_n \lambda^n,$$

where $\lambda : 1 \le |\lambda| \le 2$. Thus,

$$\sigma((f_n)_n) = \{p_\lambda(f) : |\lambda| \in [1,2]\} = \{\sum_{n=-\infty}^{\infty} f_n \lambda^n, |\lambda| \in [1,2]\}.$$

(9) Let $\{\mathscr{A}\}$ be a Banach algebra and denote by $\rho_{\mathscr{A}}$ the (open) set of its invertible elements. Let $x \in \partial\rho_{\mathscr{A}}$, the boundary of $\rho_{\mathscr{A}}$ and $x_n \in \rho_{\mathscr{A}}, x_n \to x$. Prove that $\|x_n^{-1}\| \to \infty$.
Hint: Argue by contradiction - show that x is invertible, by considering the element $e - x_n^{-1}x = x_n^{-1}(x_n - x)$. Clearly x cannot be invertible (why?).
Solution:
Assume for a contradiction that for some subsequence, $\|x_{n_k}^{-1}\| \le M$. Then, since

$$\|x_{n_k}^{-1}(x_{n_k} - x)\| \le M\|x_{n_k} - x\| \to 0,$$

as $k \to \infty$, it will follow that

$$\|e - x_{n_k}^{-1}x\| \to 0,$$

and hence $\|e - x_{n_k}^{-1}x\| < \frac{1}{2}$ for all large enough k. Thus,

$$x_{n_k}^{-1}x = e - (e - x_{n_k}^{-1}x),$$

will be invertible by the von Neumann series, whence $x = x_{n_k}(x_{n_k}^{-1}x)$ is invertible as well. However, the sets of invertible elements form an open set and in particular an invertible element has a whole neighborhood around it, composed of invertible elements. Thus x cannot be on the boundary $\partial\rho_{\mathscr{A}}$ (since each neighborhood will contain non-invertible elements), a contradiction. In fact, from the argument above, we may derive the quantitative statement, namely for every invertible element $x \in \mathscr{A}$, one has

$$\|x^{-1}\| \ge \frac{1}{dist(x, \partial\rho_{\mathscr{A}})}.$$

In the situation above

$$\|x_n^{-1}\| \ge \frac{1}{\|x_n - x\|}.$$