SOLUTION OF SOME PROBLEMS FROM HOMEWORK SET I

Problem 46/page 26

By assumption $u \in C^2$, hence $h \in C^1$. Also, there is the equality $u_{xy} = u_{yx}$. Now, we need to check Cauchy-Riemann. We have

$$\begin{aligned}(\Re h)_x &= u_{yx} = u_{xy} = (\Im h)_y \\ (\Re h)_y &= u_{yy} = -u_{xx} = -(\Im h)_x.\end{aligned}$$

Problem 53/page 27
Let $f = F' = F_x = u_x + iv_x$. Note that by Cauchy-Riemann $u_x = v_y$, $u_y = -v_x$.

$$\int_0^{2\pi} f(e^{it})e^{it}dt = \int_0^{2\pi}(u_x\cos(t) - v_x\sin(t))dt + i\int_0^{2\pi} u_x\sin(t) + v_x\cos(t)dt$$

Now

$$\begin{aligned}\int_0^{2\pi}(u_x\cos(t) - v_x\sin(t))dt &= -\int_0^{2\pi}(v_y\cos(t) + v_x\sin(t))dt = \\ &= \int_0^{2\pi}\frac{d}{dt}[v(\cos(t),\sin(t))]dt = 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_0^{2\pi} u_x\sin(t) + v_x\cos(t)dt &= \int_0^{2\pi} u_x\sin(t) - u_y\cos(t) = \\ &= -\int_0^{2\pi}\frac{d}{dt}[u(\cos(t),\sin(t))]dt = 0.\end{aligned}$$

Problem 56/page 27
Define the functions as in the Hint. Define

$$\tilde{F}(z) = \begin{cases} F_j(z) - c_{j,1} & z \in U_j, j \geq 2 \\ F_1(z) & z \in U_1 \end{cases}$$

Clearly, $\tilde{F}(z)$ is analytic in each U_j and hence in $\bigcup U_j$. Thus, we only need to show that the definition is consistent.

Next, let $1 \leq j_1 < j_2$ and $z \in U_{j_1}$. We need to show that $F_{j_1}(z) - c_{j_1,1} = F_{j_2}(z) - c_{j_2,1}$ for $z \in U_{j_1}$. For this values of z, $F_{j_2}(z) - F_{j_1}(z) = c_{j_2,j_1}$. Thus, matters reduce to showing that

$$c_{j_2,j_1} + c_{j_1,1} = c_{j_2,1}. \tag{1}$$

This last statement is true, since for $z \in U_1 \subseteq U_{j_1}$

$$F_{j_2}(z) - F_{j_1}(z) = c_{j_2,j_1}; F_{j_1}(z) - F_1(z) = c_{j_1,1};\ F_{j_2}(z) - F_1(z) = c_{j_2,1}$$

Adding the first two on the left yields the third left hand side, hence (1).

Last problem in the assignment:

Let $F = u + iv$ is analytic. By the Cauchy-Riemann, we have that $u_x = v_y$. Thus, applying the theorem 1.5.1 (for simply connected domains), we can construct w, so that

$$w_y = u, w_x = v.$$

Hence, $F = u + iv = w_y + iw_x$. In addition, by the second Cauchy-Riemann

$$w_{xx} = v_x = -u_y = -w_{yy},$$

whence w is harmonic. This is clearly unique (up to a constant). Indeed, assuming another $\tilde{w}$ with this properties, it follows that $\tilde{w}_x = w_x$ and $\tilde{w}_y = w_y$, whence $\tilde{w} = w + const.$.

The statement for z follows in a similar manner (in fact, one can take z to be the conjugate harmonic function of w.)