On the stability of standing waves for $\mathcal {PT}$ symmetric Schrödinger and Klein-Gordon equations in higher space dimensions

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2 authors:

Milena Stanislavova
University of Kansas

A. Stefanov
University of Kansas

ON THE STABILITY OF STANDING WAVES FOR $\mathcal{PT}$ SYMMETRIC SCRÖDINGER AND KLEIN-GORDON EQUATIONS IN HIGHER SPACE DIMENSIONS

MILENA STANISLAVOVA AND ATANAS STEFANOV

ABSTRACT. We consider $\mathcal{PT}$ Schrödinger and Klein-Gordon equations in higher dimensional spaces. After the construction of the standing waves, we proceed to study their spectral stability. This extends, in the Schrödinger case, the recent results of Alexeeva et. al. [1] and Bludov et. al. [6].

1. INTRODUCTION

1.1. **Motivation, some history and previous works.** About twenty years ago, Bender and Boettcher, [3] have observed, on a very specific model, that parity and time symmetries can create purely real spectrum of otherwise non - selfadjoint operators. On the mathematical side, significant progress was made in [19, 20] where a characterization of such non-self adjoint Hamiltonians with real spectrum was found. While examples of such kind have existed before that, the Bender-Boettcher work has been influential in that it spurred numerous studies, where concrete physical applications were found, especially in quantum mechanics and waveguide optics. In fact, the explicit Bender-Boettcher potential turned out to be relevant in the study of superconducting wires, [24]. Note that all these early studies have been concentrated on the linear aspects of the theory - that is of interest was the behavior of a *linear Hamiltonian system* in the form $i\psi_t = \mathcal{H}\psi$, where $H^* \neq H$, but H is special in that it commutes with a $\mathcal{PT}$ operator, that is $\mathcal{PT}$-symmetric Hamiltonian. Later on, many researchers have considered actual nonlinear models, driven by $\mathcal{PT}$-symmetric Hamiltonians. We will not give here an extensive review of these developments, but we refer the reader to the excellent recent review article, [17]. From a modeling perspective, the easiest (non-selfadjoint) model to consider is of the form[1]

$$\left| \begin{array}{l} ix_1' = -i\gamma x_1 + \kappa x_2 \\ ix_2' = \kappa x_1 + i\gamma x_2 \end{array} \right.$$

Adding the usual Kerr type interaction (and taking the same leading order approximation) leads to the $\mathcal{PT}$ symmetric dimer, namely

$$\left| \begin{array}{l} ix_1' = -i\gamma x_1 + \kappa x_2 + c|x_1|^2 x_1 \\ ix_2' = \kappa x_1 + i\gamma x_2 + c|x_2|^2 x_2 \end{array} \right. \tag{1}$$

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Milena Stanislavova Department of Mathematics, University of Kansas, 1460 Jayhawk Boulevard, Lawrence KS 66045–7523; stanis@ku.edu.

Atanas Stefanov Department of Mathematics, University of Kansas, 1460 Jayhawk Boulevard, Lawrence KS 66045–7523; stefanov@ku.edu.

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[1] see (36), (37) in [17] - note that this is a leading order approximation for stationary propagation of light in an optical coupler with gain and loss, [7]

The $\mathcal{PT}$ symmetric dimer is a typical non-linear $\mathcal{PT}$-symmetric model, which was extensively studied in the literature, [28]. Further studies have considered generalizations of such models, namely discrete $\mathcal{PT}$-symmetric networks, [23]. These systems feature finitely many variables $x_1, \ldots, x_n$, where the linear part involves closest neighbor interactions only. A consideration of a large number of spatial nodes, with small distance between them, naturally leads to a continuous model with two coupled nonlinear Schrödinger equations with gain and loss, namely (see (107) in [17])

$$\left| \begin{array}{l} iu_t = -u_{xx} - \kappa v + i\gamma u + (c_1|u|^2 + c_2|v|^2)u \\ iv_t = -v_{xx} - \kappa u - i\gamma v + (c_2|u|^2 + c_1|v|^2)v \end{array} \right. \tag{2}$$

where $\kappa \geq 0, \gamma \geq 0$. This model was analyzed in detail in series of recent works, [1], [2] and a more general version is considered in [6]. More precisely, the authors in these papers have explicitly constructed such waves (in the form (5) below) and in addition, they have studied their linearized stability properties. We will not review their results in detail, since the goal of this paper is to generalize them.

We do so by extending these results in several directions. First, we consider more general coupling, namely the Kerr interaction potentials are replaced by a general power p functions, which are physically relevant in certain regimes[2]. Next, we consider these models in their higher dimensional form, which substantially departs from the available results in the literature. It turns out however that this fits well within our methods and no essentially new techniques are needed in the study of these models. Next, we discuss the specific models under consideration.

1.2. **The models and the solitons.** We consider the model of $\mathcal{PT}$ coupled Schrödinger equations. Our presentation, for the Schrdinger case, mostly follows the recent works [1], [6], where the one dimensional case was studied in detail. More specifically, for a parameter $\alpha > -1$, we consider the Schrödinger version

$$\left\{ \begin{array}{l} iu_t + \Delta u + (|u|^{p-1} + \alpha|v|^{p-1})u + v = i\gamma u \\ iv_t + \Delta v + (\alpha|u|^{p-1} + |v|^{p-1})v + u = -i\gamma v \end{array} \right. \quad u, v : \mathbf{R}^1_+ \times \mathbf{R}^d \to \mathcal{C} \tag{3}$$

and the Klein-Gordon equations

$$\left\{ \begin{array}{l} u_{tt} - \Delta u + u - (|u|^{p-1} + \alpha|v|^{p-1})u + v = i\gamma u \\ v_{tt} - \Delta v + v - (\alpha|u|^{p-1} + |v|^{p-1})v + u = -i\gamma v \end{array} \right. \tag{4}$$

Our main object of study will be the existence and stability of solitary wave solutions for (3) and (4). More precisely, introduce the variables

$$u = e^{i\omega t} e^{-i\theta} U, v = e^{i\omega t} V, \tag{5}$$

In terms of U, V, the Schrödinger $\mathcal{PT}$ system (3) becomes

$$\left\{ \begin{array}{l} iU_t + \Delta U - \omega U + (|U|^{p-1} + \alpha|V|^{p-1})U = -\cos(\theta)V + i(\gamma U - \sin(\theta)V) \\ iV_t + \Delta V - \omega V + (\alpha|U|^{p-1} + |V|^{p-1})V = -\cos(\theta)U + i(\sin(\theta)U - \gamma V). \end{array} \right. \tag{6}$$

while for the Klein-Gordon equation, we obtain

$$\left\{ \begin{array}{l} U_{tt} + 2i\omega U_t - \Delta U + (1-\omega^2)U - (|U|^{p-1} + \alpha|V|^{p-1})U = -\cos(\theta)V + i(\gamma U - \sin(\theta)V) \\ V_{tt} + 2i\omega V_t - \Delta V + (1-\omega^2)V - (\alpha|U|^{p-1} + |V|^{p-1})V = -\cos(\theta)U + i(\sin(\theta)U - \gamma V) \end{array} \right. \tag{7}$$

As in previous works, [1, 6], one takes $\gamma = \sin(\theta)$. As a result, if we look for stationary, positive and decaying solutions of (6) in the form $U = V = \phi$, we arrive at the single equation

$$-\Delta\phi + a^2\phi - (1+\alpha)\phi^p = 0, \phi : \mathbf{R}^d \to \mathbf{R}^1. \tag{8}$$

[2] see [17] for a discussion

where we have denoted for convenience $a^2 = \omega - \cos(\theta)$. Thus implicitly, when dealing with this type of solitons, we will require that $\omega > \cos(\theta)$ and $\alpha > -1$. Similarly, stationary, positive and decaying solutions of (7), with $U = V = \psi$, are given by

$$-\Delta\psi + b^2\psi - (1+\alpha)\psi^p = 0, \psi : \mathbf{R}^d \to \mathbf{R}^1, \tag{9}$$

where we have denoted $b^2 = 1 - \omega^2 + \cos(\theta)$. Similarly, we will implicitly require that $1 + \cos(\theta) > \omega^2$.

The equation (8) is well-studied in the literature. Existence of solutions was established by variational methods in [5], see also [4]. These were the so-called ground state solutions, which are obtained by constrained minimization methods. More specifically, for $p \in (1, p_{\max})$,

$$p_{\max} = \begin{cases} 1 + \frac{4}{d-2} & d \geq 3 \\ \infty & d = 1, 2, \end{cases}$$

there are positive and decaying solutions of (8). Establishing the uniqueness of such solutions proved to be much harder problem - it was studied in [9], for a particular case and then in [18] for the full range of p as in the existence results. More precisely, it was shown that for $p \in (1, p_{\max})$, the equation (8) has an unique positive and decaying solution, modulo the translation invariances. That is, there is an unique bell-shaped[3] and decaying function, say $\varphi_{p,d} : \mathbf{R}^1 \to \mathbf{R}^1_+$, solving

$$-\Delta\varphi_{p,d} + \varphi_{p,d} - \varphi_{p,d}^p = 0. \tag{10}$$

We consider solutions of ϕ of (8), in the form

$$\phi(x) = \left(\frac{a^2}{1+\alpha}\right)^{\frac{2}{p-1}} \varphi_{p,d}(a|x - x_0|). \tag{11}$$

where $x_0 \in \mathbf{R}^d$ is arbitrary. Similarly, there exists $y_0 \in \mathbf{R}^d$, so that every solution of (9) is in the form

$$\psi(x) = \left(\frac{b^2}{1+\alpha}\right)^{\frac{2}{p-1}} \varphi_{p,d}(b|x - y_0|). \tag{12}$$

Our interest in this paper is the spectral stability of these ground state solutions for the classical Schrödinger and Klein-Gordon models, but in the framework of the $\mathcal{PT}$ symmetric versions (6) and (7). Clearly, the stability of these solitons is independent on the translations x_0, y_0, so we take $x_0 = 0$ in (11) and $y_0 = 0$ in (12).

1.3. **The linearization and statement of main results.** In this section, we provide the rigorous framework for the stability of the solitary waves constructed in the previous section, both in the Schrödinger and the Klein-Gordon contexts.

1.3.1. *The linearization around the soliton of the Schrödinger $\mathcal{PT}$ symmetric system.* For the Schrödinger system (6), we linearize by introducing the ansatz $U = \phi + z, V = \phi + w$, which we apply in (6). Recall $\gamma = \sin(\theta)$. Ignoring the contributions of all terms in the form $O(z^2), O(w^2)$, we arrive at the following system

$$\begin{aligned} &iz_t + \Delta z - \omega z + (1+\alpha)\phi^{p-1}z + (p-1)\phi^{p-1}\Re z + \alpha(p-1)\phi^{p-1}\Re w = -\cos(\theta)w + i\gamma(z - w) \\ &iw_t + \Delta w - \omega w + (1+\alpha)\phi^{p-1}w + (p-1)\phi^{p-1}\Re w + \alpha(p-1)\phi^{p-1}\Re z = -\cos(\theta)z + i\gamma(z - w) \end{aligned}$$

Following [1], we introduce the new variables,

$$r_1 + ir_2 = r = z + w \quad s_1 + is_2 = s = z - w$$

[3] i.e. even, positive and decreasing in $[0, \infty)$

In these variables, the eigenvalue problem takes the form

$$\left\{\begin{array}{l} -r_2' + \Delta r_1 - \omega r_1 + p(1+\alpha)\phi^{p-1} r_1 = -\cos(\theta) r_1 - 2\gamma s_2 \\ r_1' + \Delta r_2 - \omega r_2 + (1+\alpha)\phi^{p-1} r_2 = -\cos(\theta) r_2 + 2\gamma s_1 \\ -s_2' + \Delta s_1 - \omega s_1 + [p(1-\alpha)+2\alpha]\phi^{p-1} s_1 = \cos(\theta) s_1 \\ s_1' + \Delta s_2 - \omega s_2 + (1+\alpha)\phi^{p-1} s_2 = \cos(\theta) s_2. \end{array}\right. \tag{13}$$

Taking into account (11) (note that $x_0 = 0$), it is clear that a dilation by a factor of a^2 will simplify matters. Slightly abusing the notations, we replace $r \to e^{a^2 t\mu} r(a\cdot), s \to e^{a^2 t\mu} s(a\cdot)$ to rewrite the problem as follows

$$\mathcal{J}\mathcal{L}\vec{X} = \mu \vec{X}, \vec{X} = \begin{pmatrix} r_1 \\ r_2 \\ s_1 \\ s_2 \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\theta,a} = \begin{pmatrix} L & \Gamma_a J \\ \mathbf{0}_2 & \tilde{L} + \eta_a \end{pmatrix}, \ \mathcal{J} = \begin{pmatrix} J & \mathbf{0}_2 \\ \mathbf{0}_2 & J \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ L &= \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \tilde{L} = \begin{pmatrix} L_\alpha & 0 \\ 0 & L_- \end{pmatrix}, \ \Gamma_a = -\frac{2\gamma}{a^2}, \ \eta_a = \frac{2\cos(\theta)}{a^2}. \end{aligned}$$

and the scalar operators $L_\pm$ are given by

$$\begin{aligned} L_+ &= -\Delta + 1 - p\varphi^{p-1}; L_- = -\Delta + 1 - \varphi^{p-1}, \\ L_\alpha &= -\Delta + 1 - \frac{p(1-\alpha)+2\alpha}{1+\alpha}\varphi^{p-1} =: -\Delta + 1 - p_\alpha \varphi^{p-1}. \end{aligned}$$

Here we have used the short cut $\varphi = \varphi_{p,d}$ for the unique ground state solution of (10). We will also drop the subscripts from the notations Γ_a, η_a and we will instead use only Γ, η.

It is clear from the form of the eigenvalue problem (14) that the spectral properties of $L_\pm$ play an important role. Clearly, by Weyl's theorem

$$\sigma_{a.c.}(L_+) = \sigma_{a.c.}(L_-) = \sigma_{a.c.}(L_\alpha) = [1, \infty).$$

Regarding the point spectrum, it was proved in [18], see also [29], that $L_- \geq 0$, while L_+ has a simple negative eigenvalue. The kernels of both operators also admit an explicit description. More concisely[4]

$$L_- \geq 0, \quad L_-[\varphi] = 0, \quad L_-|_{\{\varphi\}^\perp} \geq \kappa^2 > 0, \tag{15}$$

$$L_+[\phi_0] = \sigma_{p,d}\phi_0, \quad Ker[L_+] = span[\frac{\partial \varphi}{\partial_{x_j}} : j = 1, \ldots, d], \quad L_+|_{\{\phi_0, \nabla\varphi\}^\perp} \geq \kappa^2 > 0 \tag{16}$$

Regarding the operator L_α, an elementary algebra shows that $p_\alpha > p$ for $\alpha \in (-1, 0)$, $p_\alpha \in (1, p)$ for $\alpha \in (0, 1)$ and $p_\alpha < 1$ for $\alpha > 1$. By obvious comparisons with $L_\pm$ and (15), (16), it follows that

$$L_\alpha > 0, \alpha \in (1, \infty) \tag{17}$$

$$n(L_\alpha) = 1, \alpha \in (0, 1) \tag{18}$$

$$n(L_\alpha) \geq d + 1, \alpha \in (-1, 0), \tag{19}$$

[4]The value of $\sigma_{p,d} < 0$ is in general not known explicitly except in $d = 1$, in which case $\sigma_{p,1} = 1 - \frac{(p+1)^2}{4}$.

where we have used the notation $n(S)$ to denote the number of negative eigenvalues of a self-adjoint operator S. Denote the bottom of the spectrum of L_α by σ_α, a simple eigenvalue. As a consequence of (17), (18), $\sigma_\alpha \in (0,1)$ if[5] $\alpha > 1$ and $\sigma_\alpha \in (\sigma_{p,d}, 0)$, if $\alpha \in (0,1)$.

1.3.2. *The linearization around the soliton of the Klein-Gordon $\mathcal{PT}$ symmetric system.* For the Klein-Gordon equation, we apply similar approach. Apply the ansatz $U = \psi + z_1 + iz_2, V = \psi + w_1 + iw_2$ in the linearized equation (7) (again recalling $\gamma = \sin(\theta)$), we obtain

$$\left\{ \begin{array}{l} z_1'' - 2\omega z_2' - \Delta z_1 + (1-\omega^2) z_1 - 2p(1+\alpha)\psi^{p-1} z_1 = -\cos(\theta) w_1 - \gamma(z_2 - w_2) \\ z_2'' + 2\omega z_1' - \Delta z_2 + (1-\omega^2) z_2 - 2(1+\alpha)\psi^{p-1} z_2 = -\cos(\theta) w_2 + \gamma(z_1 - w_1) \\ w_1'' - 2\omega w_2' - \Delta w_1 + (1-\omega^2) w_1 - 2(1+\alpha)p\psi^{p-1} w_1 = -\cos(\theta) z_1 - \gamma(z_2 - w_2) \\ w_2'' + 2\omega w_1' - \Delta w_2 + (1-\omega^2) w_2 - 2(1+\alpha)\psi^{p-1} w_2 = -\cos(\theta) z_2 + \gamma(z_1 - w_1). \end{array} \right. \tag{20}$$

Looking at the form of the solution ψ in (12), a scaling by b will simplify matters. In addition, it is again advantageous to pass to the variables $(Z, W) \to (Z+W, Z-W)$. We combine both in one change of variables. Namely, take

$$\begin{array}{rcl} r_1 &=& e^{b^2 t\mu}[z_1(b\cdot) + w_1(b\cdot)], r_2 = e^{b^2 t\mu}[z_1(b\cdot) - w_1(b\cdot)], \\ s_1 &=& e^{b^2 t\mu}[z_2(b\cdot) + w_2(b\cdot)], s_2 = e^{b^2 t\mu}[z_2(b\cdot) - w_2(b\cdot)]. \end{array}$$

which allows us to rewrite the linearized system (15) in the form

$$\mu^2 \vec{X} + 2\mu\omega \mathcal{J}\vec{X} + \mathcal{L}\vec{X} = 0, \quad \vec{X} = \left(\begin{array}{c} r_1 \\ r_2 \\ s_1 \\ s_2 \end{array} \right), \tag{21}$$

with the same notations as in the Schrödinger eigenvalue problem (14), with the only exception being that the parameter a is replaced by b. In other words,

$$\Gamma_b := -\frac{2\gamma}{b^2}, \eta_b := \frac{2\cos(\theta)}{b^2}.$$

Thus, the eigenvalue problems under consideration will be in the form (14) and (21).

1.4. **Main results.** The next result gives stability/instability results for general dimension $d \geq 1$ for the $\mathcal{PT}$ symmetric Schrödinger model.

Theorem 1. *Let θ, ω be so that $\omega > \cos(\theta)$ and $\eta_a = \eta_{\omega,\theta} = \frac{2\cos(\theta)}{\omega - \cos(\theta)}$. Consider the waves $(e^{i\omega t} e^{-i\theta}\phi, e^{i\omega t}\phi)$, with ϕ given by (11). Then, if $p > 1 + \frac{4}{d}$, the waves are spectrally unstable with a real growing mode. In the remaining cases, assume that $1 < p \leq 1 + \frac{4}{d}$. Then,*

1. *If $\alpha > 1$, then the waves are spectrally stable, if $\eta \geq 0$ and spectrally unstable with at least one single real growing mode, if $\eta \in (-\sigma_\alpha, 0)$.*
2. *If $\alpha = 1$, then the waves are stable for all η.*
3. *If $\alpha \in [0,1)$, then the waves are spectrally stable, if $\eta \geq -\sigma_\alpha$ and unstable with a real growing mode, if $\eta \in (0, -\sigma_\alpha)$.*
4. *If $\alpha \in (-1, 0)$ and $\eta > -\sigma_\alpha$, then the waves are spectrally stable. If $\eta \in (0, -\sigma_\alpha)$, then the waves are unstable. If[6] $\eta \in (-\lambda_1(L_-), 0)$, then the waves are unstable with at least d unstable real eigenvalues.*

Some remarks are in order:

[5] $\sigma_\alpha < 1$, since L_α does not have embedded eigenvalues in $\sigma_{ac}(L_\alpha) = [1, \infty)$

[6] Here $\lambda_1(L_-) = \inf_{\|h\|=1: h \perp \varphi} \langle L_- h, h \rangle > 0$ is the second smallest eigenvalue of L_-, if any. If L_- does not support any other eigenvalues beyond zero, $\lambda_1(L_-) = 1$

(1) The case $\eta < -1$ is a difficult one to investigate, in part due to the failure of the gap condition, that is $0 \in \sigma_{ess.}(\mathcal{J}(\tilde{L}+\eta))$. Several numerical studies in the one dimensional case ([10, 1], see also [22]) predict that oscillatory instabilities will appear. On the other hand, the only rigorous work that we are aware of, [22], establishes the appearance of a single quartet of eigenvalues, in the limit $\eta \to -\infty$, again for $d = 1$. A natural outstanding open question then is to show the instability of such waves in the higher dimensional context $d \geq 2$, as well as the case $\eta < -1$, where $|\eta|$ is not necessarily large.
(2) The case $\alpha = 1$ presents itself as an interesting bifurcation point. It would be interesting to see how the stability for all values of η reconciles with the instabilities for various values of η in the cases $1+\epsilon > \alpha > 1$ and $1-\epsilon < \alpha < 1$. The results seem to suggest that there is stability for example, if $\alpha \in (1, 1+\epsilon)$ and $\eta \in (-1, -\sigma_\alpha)$.
(3) Related to the previous point, it is unclear at this point, how to treat the cases $\alpha > 1$ and $\eta \in (-1, -\sigma_\alpha)$ as well as $\alpha \in [0, 1)$, $\eta \in (-1, 0)$. It is expected that these configurations will be unstable, but we are not able to use the index counting theories.

Our next result concerns the Klein-Gordon problem. Again, we provide some basic information regarding the stability of the standing waves in the standard Klein-Gordon model. This will provide the context in which we study the stability for these waves in the $\mathcal{PT}$ situation. In order to state the relevant stability results, we need to define

$$\omega_{p,d} := \sqrt{\frac{p-1}{4-(p-1)(d-1)}}.$$

for $p > 1, d \geq 1$ and $p < 1 + \frac{4}{d}$. Note that $\omega_{p,d} \in (0, 1)$, when $p < 1 + \frac{4}{d}$. Then, the waves $e^{i\omega t}\psi$ are unstable for the Klein-Gordon model, if and only if $1 + \frac{4}{d} \leq p < p_{\max}$ or otherwise $1 < p < 1 + \frac{4}{d}$ and $|\omega| < \omega_{p,d}$, see also Proposition 1 below.

We are now ready for our main result concerning the eigenvalue problem (21).

Theorem 2. *Let ϕ be given by (12) and $\omega : \omega^2 < 1 + \cos(\theta)$, $\eta_b = \frac{2\cos(\theta)}{b^2} = \frac{2\cos(\theta)}{1-\omega^2+\cos(\theta)}$. Then, if $1 + \frac{4}{d} \leq p < p_{\max}$ or $1 < p < 1 + \frac{4}{d}$, but $|\omega| < \omega_{p,d}$, then the waves are unstable.*

Assuming $1 < p < 1 + \frac{4}{d}$, $|\omega| > \omega_{p,d}$, we have

(1) *For $\alpha > 1$ and $\eta > 0$, the waves are stable, while for $\eta \in (-\sigma_\alpha, 0)$, the waves are unstable, with a real growing mode.*
(2) *For $\alpha = 1$, the waves are stable for $\eta \geq -\omega^2$. For $\eta < -\omega^2$, we have oscillatory instability with eigenmodes in the form $\sqrt{-(\eta+\omega^2)} \pm i\omega$.*
(3) *For $\alpha \in [0, 1)$ and $\eta > -\sigma_\alpha$, then the waves are spectrally stable. If $\eta \in (0, -\sigma_\alpha)$, then the waves are unstable.*
(4) *If $\alpha \in (-1, 0)$ and $\eta > -\sigma_\alpha$, then the waves are spectrally stable. If $\eta \in (0, -\sigma_\alpha)$, then the waves are unstable. If $\eta \in (-\lambda_1(L_-), 0)$, then the waves are unstable with at least d positive real eigenvalues.*

The open problems that we have outlined after Theorem 1 largely apply here. Interestingly, the case $\alpha = 1$ presents itself differently for the Klein-Gordon models.

2. Spectral stability of standing waves for the $\mathcal{PT}$ symmetric Schrödinger model: Proof of Theorem 1

As we have discussed in the introduction, we base our stability arguments on the Hamilton-Krein index theory. Some elements of this approach were present in the early pioneering works of Grillakis-Shatah-Strauss, [13, 11, 12] and Weinstein, [29], but we follow the more systematic approach developed by Kapitula-Kevrekidis-Sandstede [14, 15] (see also [16]) and alternatively in [8].

2.1. **Some basic Hamilton-Krein instability index theory.** More precisely, for a self-adjoint operator K with finitely many negative eigenvalues, consider an eigenvalue problem of the form

$$\tilde{J}Kf = \mu f. \tag{22}$$

where $\tilde{J}^* = -\tilde{J}, K^* = K$ and some mild additional technical assumptions on $\tilde{J}, K$, which are easily met in our case. Assume also $\Im K = 0$, meaning that K maps real-valued into real-valued functions. One is interested in the "instabilities count" for the eigenvalue problem (22). In other words, how many solutions are there (μ, f), with $f \neq 0, \Re\mu > 0$. One (almost immediate) consequence of the form (22) is that if $K \geq 0$, then (22) has no instabilities.

Next, assume that $Ker(K)$ is finite dimensional and let $Ker(K) = span\{\psi_j : j = 1, \ldots, N\}$, where $\{\psi_j\}_{j=1}^N$ are linearly independent. Assume also that $\tilde{J}$ is invertible and $\tilde{J}^{-1}[Ker[K]] \subset Ker[K]^\perp$. Introduce $D = (D_{ij})_{i,j=1}^N$ and $D_{ij} = \left\langle K^{-1}[\tilde{J}^{-1}[\psi_i]], \tilde{J}^{-1}\psi_j \right\rangle$ and assume that D is invertible. Denoting the number of different solutions $(f, \mu) : f \in D(K), \Re\mu > 0$ by $n_{unstable}(\tilde{J}K)$, we have the relation

$$n_{unstable}(\tilde{J}K) + n_{negativeKrein}(\tilde{J}K) = n(K) - n(D), \tag{23}$$

where $n_{negativeKrein}(\tilde{J}K)$ is an even number of marginally stable eigenvalues of (22) with negative Krein signature. As an immediate consequence of (23), we have $n_{unstable}(\tilde{J}K) \leq n(K) - N(D)$ and in addition, $n_{unstable}(\tilde{J}K) \geq 1$, if $n(K) - n(D)$ is odd. In the particular case when $n(K) = 1$, the eigenvalue problem (22) has exactly one real instability if all eigenvalues of the symmetric matrix D are positive. Otherwise, D has exactly one negative eigenvalue and the eigenvalue problem (22) is stable. We note that the non-solvability of (22) for specific μ means that the operator $\tilde{J}K - \mu$ is an invertible.

Another useful result is an easy corollary of Remark 3.1 in [14], which has appeared earlier in [11]. It states that if the eigenvalue problem (22) is in the form $\tilde{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, K = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$, with $D > 0$ and $Ker[K] = \{0\}$, then one has

$$n_{real\ instabilities}(\tilde{J}K) \geq |n(L_2) - n(L_1)|. \tag{24}$$

2.2. **Proof of Theorem 1.** In this section, we prove Theorem 1. We work with the eigenvalue problem (14). Since $\mathcal{J}^{-1} = -\mathcal{J}$, we can rewrite it as

$$\mathcal{L}\vec{X} = -\mu\mathcal{J}\vec{X}.$$

Next, we exploit the upper triangular structure of $\mathcal{L}$. More specifically, letting $\vec{X} = \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix}$, we have the system

$$L\mathbf{Y} + \Gamma J\mathbf{Z} = -\mu J\mathbf{Y} \tag{25}$$

$$(\tilde{L} + \eta Id)\mathbf{Z} = -\mu J\mathbf{Z}. \tag{26}$$

The first equation can be rewritten[7], using $J^2 = -Id$,

$$(JL - \mu)\mathbf{Y} = \Gamma\mathbf{Z}. \tag{27}$$

[7]From this point on, for the brevity of the notation, we will drop the Id from our operator notations. Namely whenever we write $T + \mu$, where T is an operator and μ is a scalar, we mean $T + \mu Id$

One immediately recognizes that the associated homogeneous eigenvalue problem

$$JL\mathbf{Y}_0 = \mu \mathbf{Y}_0, \tag{28}$$

is exactly the eigenvalue problem for the solitary wave ϕ as a solution to the standard semi-linear Schrödinger equation. We claim that this immediately implies instability for (25), (26), if we have started with an unstable wave ϕ for the ordinary Schrödinger system, that is for $p > 1 + \frac{4}{d}$.

Indeed, if $p > 1 + \frac{4}{d}$, it is known that one has a real simple instability, say $\mu_0 > 0$. That is, there is $\mathbf{Y}_0 \neq 0$, so that $(JL - \mu_0)\mathbf{Y}_0 = 0$. Clearly then, $\mathbf{Z} = 0$, $\mathbf{Y}_0$ and $\mu = \mu_0$ provide a non-trivial solution of (25), (26), whence instability is established.

Assume now that $1 < p \leq 1 + \frac{4}{d}$. From the (spectral) stability of the solitary wave for the standard Schrödinger model, it follows that $JL - \mu$ is invertible, whenever $\mu \notin i\mathbf{R}^1$. So, take $\mu \notin i\mathbf{R}^1$. We can solve (27) (or equivalently (25)) by using the formula

$$\mathbf{Y} = \Gamma(JL - \mu)^{-1}[\mathbf{Z}]. \tag{29}$$

Thus, it remains to concentrate on the study of the eigenvalue problem (26), which is only in terms of $\mathbf{Z}$. If (26) has a non-trivial solution $\mathbf{Z}$ for some $\mu \notin i\mathbf{R}^1$, we conclude instability[8] and stability otherwise. Again, applying J on both sides of (26), matters are reduced to

$$J(\tilde{L} + \eta)\mathbf{Z} = \mu \mathbf{Z}. \tag{30}$$

The eigenvalue problem (30) is mostly amenable to the methods of the Hamilton-Krein index theory, as explained in Section 2.1. To that end, note that $J^* = -J$, while $\tilde{L} + \eta Id$ is a self-adjoint, bounded from below operator. We should mention here that an eigenvalue problem in the form (30) was already considered, in the work of Pelinovsky, [21]. Herein, we chose instead to follow the simpler approach outlined in Section 2.1. The reason is that in all cases[9] the definite predictions can be obtained by either method, while in the inconclusive cases, both approaches fail to produce a definite result.

Let $\alpha > 1$, so $\sigma_\alpha > 0$. Then, $L_\alpha \geq \sigma_\alpha Id$. Clearly, if $\eta \geq 0$, we have spectral stability due to the fact that $\tilde{L} + \eta \geq 0$. If $\eta \in (-\sigma_\alpha, 0)$, we have that $n(\tilde{L} + \eta) = 1$, whence by (23), we conclude instability, with a single real eigenvalue. The case $\eta \in (-\infty, -\sigma_\alpha)$ is open, since our method is inconclusive for this configuration.

For $\alpha = 1$, we have $L_\alpha = L_-$, so the eigenvalue problem (30) reduces to

$$\left| \begin{array}{l} (L_- + \eta) z_1 = -\mu z_2 \\ (L_- + \eta) z_2 = \mu z_1. \end{array} \right.$$

This is equivalent to $(L_- + \eta)^2 z_1 = -\mu^2 z_1$, whence $\mu \in i\mathbf{R}^1$, since $(L_- + \eta)^2 \geq 0$. This implies stability for all values of η.

For $\alpha \in (0, 1)$, we have $\sigma_\alpha : \sigma_{p,d} < \sigma_\alpha < 0$. Clearly, for $\eta \geq -\sigma_\alpha$, we have $\tilde{L} + \eta \geq 0$, hence stability. For $\eta \in (0, -\sigma_\alpha)$, $n(\tilde{L} + \eta) = 1$, hence instability, with a positive growing mode. The case $\eta < 0$ is open, as our approach does not give a definite prediction about the stability. The case $\alpha = 0$ is covered by the same argument, once we observe that $L_\alpha = L_+$ and $\sigma_\alpha = \sigma_{p,d}$.

If $\alpha \in (-1, 0)$ and $\eta > -\sigma_\alpha$, we have again $\tilde{L} + \eta \geq 0$, hence stability. If $\eta \in (0, -\sigma_\alpha)$, we clearly have $L_- + \eta > 0$, while $n(L_\alpha) \geq 1$. By formula (24) (here $n(D) = 0$), we conclude an instability. If finally, $\eta \in (-\lambda_1(L_-), 0)$, we have $n(L_\alpha + \eta) \geq d + 1$, while $n(L_- + \eta) = 1$, we conclude by (24) (here again $n(D) = 0$) that the number of real unstable eigenvalues is at least $d + 1 - 1 = d$.

[8] since we can then solve for $\mathbf{Y}$ based on (29)

[9] except in the cases covered by (24), which may also be concluded by the results in [21]

3. Characterization of the spectral stability of the standing wave solitary waves in the Klein-Gordon case

We are now considering the eigenvalue problem (21). Introduce the auxiliary variables $\vec{X} = \begin{pmatrix} Y \\ Z \end{pmatrix}$. In terms of Y, Z, we have the following linearized system

$$\left\{ \begin{array}{l} \mu^2 Y + 2\mu\omega JY + LY = -\Gamma_b JZ \\ \mu^2 Z + 2\mu\omega JZ + (\tilde{L} + \eta_b)Z = 0 \end{array} \right. \tag{31}$$

Before we continue with our spectral analysis of (31), let us discuss the stability of the standing waves in the context of the standard Klein-Gordon system. This will have implication for the stability of the $\mathcal{PT}$ variant of the problem, namely about the existence of eigenvalues for the spectral problem (31).

3.1. The eigenvalue problem for the standard linearized Klein-Gordon equation at the ground states.

We revisit the results, which first appeared in the works of Grillakis, Shatah and Strauss, [11], [12], [13], [25], [26], [27]. More precisely, for the Klein-Gordon problem, $p \in (1, p_{\max})$

$$u_{tt} - \Delta u + u - |u|^{p-1}u = 0, x \in \mathbf{R}^d \tag{32}$$

we have ground state solutions in the form $u_{\omega,p,d} = e^{i\omega t}\varphi_{\omega,p}(x)$, where

$$-\Delta\varphi_{\omega,p,d} + (1-\omega^2)\varphi_{\omega,p,d} - \varphi_{\omega,p,d}^p = 0.$$

By the uniqueness result (presented in the discussion after (10)), $\varphi_{\omega,p,d}$ can be written as follows

$$\varphi_{\omega,p,d}(x) = (1-\omega^2)^{\frac{1}{p-1}}\varphi_{p,d}(\sqrt{1-\omega^2}x),$$

where $\varphi_{p,d}$ is the unique solution of (10). The question for stability of these waves is of course similar to the one deduced in (20). It reduces to the question of existence of $\lambda : \Re\lambda > 0$ for which

$$\lambda^2\vec{x} + 2\lambda\omega J\vec{x} + L\vec{x} = 0. \tag{33}$$

for some $0 \neq \vec{x} \in D(L)$. Then, it is well-known ([11], [12], [13], [25], [26], [27]) that

Proposition 1. *The linearized problem* (33) *is*

- *unstable, if* $1 + \frac{4}{d} \leq p < p_{\max}$.
- *unstable, if* $1 < p < 1 + \frac{4}{d}$ *and* $0 \leq |\omega| < \omega_{p,d}$.
- *stable, if* $1 < p < 1 + \frac{4}{d}$ *and* $|\omega| \geq \omega_{p,d}$.

In all cases, the instability presents itself as a simple positive eigenvalue.

Note: The stability claim is equivalent to the unique solvability of the problem

$$(\lambda^2 + 2\lambda\omega J + L)\vec{y} = R, \tag{34}$$

for all $R \in L^2 \times L^2$ for all $\lambda : \Re\lambda > 0$. In particular, stability for (33) means that (34) with $R = 0$ implies $y = 0$. On the other hand, instability means that for $R = 0$, we have a non-trivial solution $y \neq 0$ of (34). Similar to our earlier remarks for (22), if $L \geq 0$, we have stability for the pencil (34). Now that we have fully described the stability for (33), we are ready to analyze (31).

3.2. **Spectral analysis of** (31)**.** Our first observation covers the instability cases for (33). Roughly speaking, if the wave is unstable for the standard Klein-Gordon equation - then it will be unstable for the $\mathcal{PT}$ version as well.

Proposition 2. *The waves* $(e^{i\omega t}e^{-i\theta}\psi, e^{i\omega t}\psi)$ *are unstable if*

- $1+\frac{4}{d} \leq p < p_{\max}$
- $1 < p < 1+\frac{4}{d}$ *and* $|\omega| < \omega_{p,d}$.

Proof. In this cases, we know that (33) is unstable. Thus, take $Z=0$ and then $(\mu_0, Y_0) : Y_0 \neq 0$ to be the solution of

$$\mu^2 Y_0 + 2\mu\omega J Y_0 + L Y_0 = 0.$$

Such a solution, with $\mu > 0$ exists, according to the instability of (33). □

Thus, it remains to consider the case, when (33) is actually stable. That is, let $p \in (1, 1+\frac{4}{d})$ and $|\omega| > \omega_{p,d}$ and we consider the eigenvalue problem

$$\mu^2 Z + 2\mu\omega J Z + (\tilde{L}+\eta) Z = 0. \tag{35}$$

In many cases below, it will be beneficial to rewrite the eigenvalue problem (35) in the equivalent form

$$\begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & -2\omega J \end{pmatrix} \begin{pmatrix} \tilde{L}+\eta & 0 \\ 0 & \mathbf{I} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \mu \begin{pmatrix} u \\ v \end{pmatrix} \tag{36}$$

where $u = Z, v = \mu Z$.

For $\alpha > 1$ and $\eta > 0$, we have that $\tilde{L}+\eta \geq 0$, and thus, one concludes stability for (37), hence (35). Also, if $\eta \in (-\sigma_\alpha, 0)$, we have $n(\tilde{L}+\eta) = 1$, whence instability. For the case $\eta \in (-\infty, -\sigma_\alpha)$, our method is inconclusive.

We now analyze the case $\alpha = 1$. We have again the special situation, $L_\alpha = L_-$. The eigenvalue problem (35) can be considered in the variables $Z = \begin{pmatrix} f \\ g \end{pmatrix}$. We have

$$\left| \begin{array}{l} (L_- + \eta + \mu^2) f = -2\mu\omega g \\ (L_- + \eta + \mu^2) g = 2\mu\omega f. \end{array} \right.$$

whence we arrive (by applying $(L_- + \eta + \mu^2)$ to the second equation) to

$$(L_- + \eta + \mu^2)^2 f = -4\mu^2\omega^2 f. \tag{37}$$

From this, it is immediately clear that μ cannot be a real number, different than zero. Rewrite (37) in the form

$$(L_- + \eta + \mu^2 + 2i\mu\omega)(L_- + \eta + \mu^2 - 2i\mu\omega) f = 0.$$

But this is only impossible, if either $(L_- + \eta + \mu^2 - 2i\mu\omega) f = 0, f \neq 0$ or $(L_- + \eta + \mu^2 + 2i\mu\omega) h = 0$, where $h = (L_- + \eta + \mu^2 - 2i\mu\omega) f \neq 0$. So, let $\mu = a + ib$ and suppose that $(L_- + \eta + \mu^2 + 2i\mu\omega) h = 0$. Since L_- is self-adjoint, this last equality is possible if $-(\eta + \mu^2 + 2i\mu\omega)$ is in the spectrum of L_-, so in particular it is real. But

$$\mu^2 + 2i\mu\omega = (a^2 - b^2 - 2b\omega) + 2ia(b + \omega)$$

which implies that $b = -\omega$ and $-(\eta + a^2 + \omega^2) \in \sigma_{p.p.}(L_-)$. But this means that if $-(\eta + \omega^2) < 0$, then for every a, we will have $-(\eta + a^2 + \omega^2) < 0$, hence outside of $\sigma_{p.p.}(L_-)$. Conversely, if $-(\eta + \omega^2) > 0$, there would be $a \neq 0$, namely $a = \sqrt{-(\eta + \omega^2)}$, so that $-(a^2 + \eta + \omega^2) = 0 \in \sigma_{p.p.}(L_-)$, hence instability. The other possibility, $(L_- + \eta + \mu^2 - 2i\mu\omega) f = 0, f \neq 0$ is investigated in a similar manner, with the same conclusion, leading to eigenmode $\sqrt{-(\eta + \omega^2)} + i\omega$. This establishes the final result in the case $\alpha = 1$, which is that there is stability in the case $\eta \geq -\omega^2$ and instability when $\eta < -\omega^2$.

For $\alpha \in [0, 1)$, the problem is considered in view of (18). If $\eta \geq -\sigma_\alpha$, $\tilde{L} + \alpha \geq 0$ and we have stability. If $\eta \in (0, -\sigma_\alpha)$, $n(\tilde{L} + \alpha) = 1$ and there is instability.

The case $\alpha \in (-1, 0)$ and $\eta > -\sigma_\alpha$ has $\tilde{L} + \alpha \geq 0$, hence stability. For $\eta \in (0, -\sigma_\alpha)$, $n(\tilde{L}+\alpha) = 1$, whence instability. For $\eta \in (-\lambda_1(L_-), 0)$, we may argue similarly to the Schrödinger case. Indeed, $n(L_\alpha + \eta) \geq d + 1$, while $n(L_- + \eta) = 1$, we conclude by (24) (here again $n(D) = 0$) that the number of real unstable eigenvalues is at least $d + 1 - 1 = d$.

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