MATH 961: PROJECT II
DUE: OCT. 19TH, 2020

(1) Let $A : D(A) \subset H \to H$ be an unbounded symmetric operator on a complex Hilbert space, with a dense domain. Prove that the following are equivalent:
  (a) There exists $\lambda \in \mathbb{C}$ so that both $A - \lambda I, A - \bar{\lambda} I$ are surjective.
  (b) $A = A^*$, i.e. A is self-adjoint.

  From this, show that if λ is real, $\lambda \in \rho(A)$, then A is self-adjoint.
  Hint: The direction 2) implies 1) should be straightforward. For the other one, you need to show $D(A^*) \subset D(A)$. Pick $x \in D(A^*)$ and use the surjectivity of $A - \lambda$ to show that there is $y \in D(A)$ so that $(A^* - \lambda)x = (A - \lambda)y$. Proceed to show that $x = y \in D(A)$.

(2) Exercise 6.5.4/page 343

(3) Let $H = L^2[0, 2\pi] = \{f : [0, 2\pi] \to \mathbb{C} : \int_0^{2\pi} |f(x)|^2 dx < \infty\}$. As is well-known, this is isometrically equivalent, via $f = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, to $l^2 = \{(a_n) : \sum_{n=-\infty}^{\infty} |a_n|^2 < \infty\}$. Consider the unbounded operator $Af = if'$, with $D(A) = H^1[0, 2\pi] = \{f \in L^2[0, 2\pi] \int_0^{2\pi} |f'(x)|^2 dx < \infty\}$. Show that A is self-adjoint. Determine $\sigma(A)$.
  Hint:
  Show that $\langle Ax, y \rangle = -\sum_{n=-2\pi\infty}^{\infty} n x_n \bar{y}_n$.

(4) Problem 6.5.8/page 344

(5) Let $A : D(A) \subset H \to H$ be a self-adjoint operator. Show that for every integer k, A^k, with domain defined inductively as $D(A^k) = \{x \in D(A) : Ax \in D(A^{k-1})\}$ is self-adjoint as well.
  Hint: Corollary 6.3.9.