MATH 960: PROJECT V
DUE: WEDNESDAY, MAY 13^{th}, 2020

(1) On the Banach space $l^p = \{(x_n)_{n=1}^{\infty} : (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} < \infty\}$, $1 < p < \infty$, consider the left shift operator $S(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. Prove that

$$P\sigma(S) = \{\lambda : |\lambda| < 1\}, R\sigma(S) = \emptyset, C\sigma(S) = \{\lambda : |\lambda| = 1\}.$$

Hint: First show that $|\lambda| > 1$ is in the resolvent set. Then, for each $|\lambda| < 1$ construct eigenvectors. Finally, for each $\lambda : |\lambda| = 1$, show that $Ker(\lambda - S) = \{0\}$. Then, solve the system $\lambda x_1 - x_2 = f_1, \ldots, \lambda x_n - x_{n+1} = f_n, \lambda x_{n+1} - x_{n+2} = 0, \ldots$ for each finitely supported $(f_1, f_2, \ldots, f_n, 0, \ldots)$ - take $x_{n+1} = x_{n+2} = \ldots = 0$ and solve backwards.

(2) Compute $\sigma(R)$ (with its components), where $R : l^p \to l^p, 1 < p < \infty$ and $R(x_1, x_2, \ldots,) = (0, x_1, x_2, \ldots)$.
Hint: Use the previous exercise and Lemma 5.2.5/page 209.

(3) Let A be a bounded operator on a Banach space X, with $\sigma(A) \subset \mathbb{C} \setminus \mathbb{R}_-$ or $\sigma(A) \cap \{\lambda \in \mathbb{R} : \lambda \leq 0\} = \emptyset$. Define the operator $B = \sqrt{A}$. Where does its spectrum lie? Prove that $B^2 = A$.
Hint: You need a proper definition of (a branch of) the holomorphic function $\sqrt{z}$. Define it through $\ln(z)$ in $\mathbb{C} \setminus \mathbb{R}_-$.

(4) Exercise 5.2.15/page 221.

(5) Suppose that a matrix $A : \mathbb{R}^n \to \mathbb{R}^n$ is idempotent, that is $A^k = 0$ for some $k \geq 1$. Let $k_0 = \min\{k : A^k = 0\}$. Prove that $k_0 \leq n$. Show that $\sigma(A) = \{0\}$.

(6) Consider the Volterra operator $T : L^2[0, 1] \to L^2[0, 1]$, defined by

$$Tf(t) = \int_0^t f(s)ds,$$

Show that the adjoint is $T^*f(t) = \int_t^1 f(s)ds$. Is T self-adjoint? Is T normal? Prove that the operator $P = T + T^*$ is a an orthogonal projection, i.e. $P = P^*, P^2 = P$. Characterize $Im(P)$.

(7) Show that for each integer n (induction)

$$T^n f(x) = \frac{1}{(n-1)!} \int_0^t (t - s)^n f(s)ds.$$

Prove that $r_T = 0$, so $\sigma(T) = \{0\}$.
Hint: Estimate $\|T^n\|$ so that one can conclude that $\lim_n \|T^n\|^{\frac{1}{n}} = 0$.
Bonus - 5 points: Try to find the inverse $(\lambda - T)^{-1}$ for each $\lambda \neq 0$. It should be like solving a linear ODE of first order.