MATH 960: PROJECT III
DUE: APRIL 21^{st}, 2020

The next 3 problems are related.

(1) Let $Ax = \{nx_n\}$. This is an unbounded operator on $l^2 = \{(x_n)_{n=1}^\infty : \sum_{n=1}^\infty |x_n|^2 < \infty\}$, but for each $\lambda \notin \mathbb{N}$, we can define its resolvent, namely

$$R_\lambda := (A - \lambda Id)^{-1}x = \{(n-\lambda)^{-1}x_n\}_n$$

Show that $R_\lambda : l^2 \to l^2$ is compact.
Hint: The easiest way to check that an operator is compact is to show that it is approximated in norm by finite rank operators.

(2) Let $V : l^2 \to l^2$ be bounded operator, $V = V^*$. Show that the equation

$$(A + V - \lambda)f = g \in l^2 \tag{1}$$

for a given $\lambda \notin \mathbb{N}$ and $g \in l^2$, has a solution if and only if $g \perp Ker(A+V-\lambda)$.

(3) Show that if $\lambda \notin \mathbb{N}$ and $Ker(A + V - \lambda) = \{0\}$, then (1) has an unique solution - write a formula for it.
Hint: For $2), 3)$, rewrite (1) in terms of $Id+R_\lambda V$. Use the fact that $Id+R_\lambda V$ has index zero, since $R_\lambda V$ is compact.

Remark: These problems show that Fredholm techniques work well for unbounded operators as well, as long as they have compact resolvents.

(4) Let X be a Banach space. Show that $A : X \to Y$ is a finite rank operator (i.e. $dim(Im(A)) < \infty$) if and only if there exists $\{x_j^*\}_{j=1}^n \subset X^*$, $\{y_j\}_{j=1}^n \subset Y$, so that

$$Ax = \sum_{j=1}^n \langle x_j^*, x\rangle y_j.$$

Hint: Take a basis in $Im(A) \subset Y$, say $\{y_j\}_{j=1}^n$. Take a dual family $\{y_j^*\}_{j=1}^n \subset Y^*$, as in the Appendix of the notes. Show that the operator $K : Y \to Y$,

$$Ky = \sum_{j=1}^n \langle y_j^*, y\rangle y_j,$$

acts as an identity on $Im(A)$. In particular, $Ax = KAx$.

(5) Let H be a real Hilbert space, with an orthonormal basis $\{e_n\}_{n=1}^\infty$ (Recall, $x = \sum_{n=1}^\infty \langle x, e_n\rangle e_n$). Prove the following:
- Assume that K is a bounded operator and there exists $\lambda_n \to 0$ and $f_n \in H : \|f_n\| = 1$ so that

$$Kx = \sum_{n=1}^\infty \lambda_n \langle x, f_n\rangle e_n. \tag{2}$$

Prove that K is compact.

- Show the converse, namely assuming that K is compact, show that there is $f_n : \|f_n\| = 1$ and $\lambda_n \to 0$, so that (2) holds

Hint: Sufficiency is as follows: Prove that the finite rank operators

$$K_N x = \sum_{n=1}^{N} \lambda_n \langle x, f_n \rangle e_n.$$

approximate K in the operator norm, i.e. $M_N := K - K_N : \|M_N\| \to 0$. Argue by contradiction: assuming the contrary, we should have that for any $\mu_N \to 0$ (pick $\mu_N = \sup_{j \geq N} |\lambda_j|$), there should be a sequence $N_j \to \infty$, so that $\mu_{N_j}^{-1} \|M_{N_j}\| \to \infty$. By UBP, pick $x : \|x\| = 1$, so that $\mu_{N_j}^{-1} \|M_{N_j} x\| \to \infty$. See what even $\mu_{N_j}^{-1} \|M_{N_j} x\| \geq 1, j = 1, 2, \ldots$, implies for x!

For the necessity, show that

$$Kx = \sum_n \langle Kx, e_n \rangle e_n = \sum_n \langle x, K^* e_n \rangle e_n,$$

with $f_n = \frac{K^* e_n}{\|K^* e_n\|}$, $\lambda_n = \|K^* e_n\|$ will do. For the last part, you might want to use that K sends weakly convergent sequences into strongly convergent.

(6) Let $K : X \to Y$ compact, where X, Y are Banach spaces. Show that if $Im(K)$ is closed, then K is finite rank, i.e. $dim(Im(K)) < \infty$.

Hint: Recall $dim(Ker(K)) < \infty$. By the closed image theorem, it must be that

$$\|[x]\|_{X/Ker(K)} = \inf_{\xi \in Ker(K)} \|x + \xi\|_X \leq c\|Kx\|_Y.$$

Take a sequence $x_n : \|x_n\| = 1$, and then a convergent subsequence $\{Kx_{n_k}\}$. What can you say about $\{[x_{n_k}]\}$? Recall that only finite dimensional Banach spaces have compact unit balls.