MATH 960: PROJECT III
DUE: MARCH 31^{st}, 2020

(1) Show that in $l^p, 1 < p < \infty$, $x^n \rightharpoonup x$ if and only if
  (a) $\sup_n \|x^n\|_{l^p} < \infty$.
  (b) For each k, $x^n_k \to x_k$.

(2) Show that in c_0, $x^n \rightharpoonup x$ if and only if
  (a) $\sup_n \|x^n\|_{c_0} < \infty$.
  (b) For each k, $x^n_k \to x_k$.

(3) Let X be a Banach space, with a separable dual X^*. Let $\{x^*_n\}_n$ be a dense set in B_{X^*}. Show that the function

$$d(x,y) = \sum_{n=1}^{\infty} \frac{|\langle x-y, x^*_n\rangle|}{2^n}$$

defines a metric on X, which is consistent with the weak topology on the bounded sets.

In other words, let $\{x_n\}$ be a bounded sequence. Prove that $x_k \rightharpoonup x$ if and only if $\lim_k d(x_k, x) = 0$.

(4) We say that a norm in a Banach space is *locally uniformly convex (LUC)*, if for every $x : \|x\| = 1$ and $\epsilon > 0$, there exists δ, so that whenever there is $y : \|y\| = 1$ and $\|\frac{x+y}{2}\| > 1 - \delta$, then $\|x - y\| < \epsilon$.

Prove that if X has a (LUC) norm and a sequence $\{x_n\}$ satisfies $x_n : x_n \rightharpoonup x$ (i.e. weakly convergent) and $\lim_n \|x_n\| = \|x\|$, then $\lim_n \|x_n - x\|_X = 0$.

Hint: Show first that matters reduce, without loss of generality, to the case $\|x_n\| = 1 = \|x\|$.

Remarks:
- All $L^p, 1 < p < \infty$ norms are (LUC).
- This gives a a necessary and sufficient condition in (LUC) spaces for a weakly convergent sequence to be norm convergent.

(5) Let X be a Banach space, so that X^* is separable. Prove that X is separable as well.

Bomus: 5 points Clearly X separable does not imply that X^* is separable (e.g. $X = l^1, X^* = l^\infty$). Which part of the proof does not go through?

Hint: Start with a sequence $\{x^*_n\} : \|x^*_n\| = 1$, which is dense in S_{X^*}. Show that there is $\{x_n\}$ in B_X, so that $|x^*_n(x_n)| \geq \frac{1}{2}$. Prove that $Y = \overline{span[x_n]}$ is a separable subspace of X. Prove that $Y = X$ (If not, pick an element $x^* \in Y^\perp$).

(6) For the space $l^1 = (c_0)^*$, we have $\|x^*\|_{l^1} = \sup_{x:\|x\|_{c_0}} |x^*(x)|$, but the supremum may not be achieved.

Prove that the supremum is achieved for $x^* \in l^1$ (i.e. there is $x \in c_0 : \|x\| = 1 : \|x^*\| = |x^*(x)|$) if and only if x^* has finite support, i.e. there exists N, so that $x^*(n) = 0, n > N$.

(7) Prove that $K = (B_{c_0}, \mathcal{U}_{l^1})$ is not compact. That is, the unit ball of c_0, endowed with the weak topology, is not compact.
Hint: Use the previous exercise and consider $f \in l^1$, with infinite support, as a function on K.