MATH 960: PROJECT II
DUE: MARCH 3^{rd}, 2020

Banach limits

(1) Exercise 2.5.7 a), page 104.

(2) Exercise 2.5.7 c), page 104.
Hint: Do the extension by Hahn-Banach, with $p(x) = \limsup_n x_n$ (do show that it is a good functional to work with). Prove (iii) before proving (ii).

(3) Exercise 2.5.7 d), page 104.
Hint: Use the shift property.

(4) Exercise 2.5.7 e), page 104. This shows that $(l^\infty)^* \neq l^1$.
Hint: Do a direct proof, based on the properties.

(5) Let H be a Hilbert space and $B : H \times H \to \mathbb{R}$ be a bounded bilinear form, i.e. $x \to B(x,y)$ and $y \to B(x,y)$ are bounded linear functionals on H. Prove that there is an uniquely defined operator $A : H \to H$, so that

$$B(x,y) = \langle x, Ay \rangle.$$

Also prove that the norm of B : $\|B\| := \sup_{x,y:\|x\|=\|y\|=1} |B(x,y)|$ and $\|A\|$ coincide.

(6) Show that for any Banach space X

$$\|x\| = \sup_{\|l\|_{X^*}=1} |l(x)|.$$

(7) Show that $X \subseteq X^{**} = (X^*)^*$. To do that, define for every $x \in X, l \in X^*$, $x(l) := l(x)$, so one can think of $x \in X^{**}$. Prove that $\|x\|_X = \|x\|_{X^{**}}$. This shows that the map $i : X \to X^{**}$ is an isometric embedding of a Banach space into its second dual.
Hint: For the isometric embedding, you may want to use the result of the previous problem.

(8) Let X, Y be Banach spaces, so that $T : X \to Y$ is a linear map. If for every bounded linear functional $g \in Y^*$, we have that the linear functional $g \circ T \in X \to \mathbb{R}$ is bounded (i.e. $g \circ T \in X^*$), show that T must be bounded as well.
Hint: Assuming for a contradiction that T is unbounded, by the UBP, there must be $x_n \in X : \|x_n\| = 1$, $\lim_n \|Tx_n\|_Y = \infty$.