# MATH 960: PROJECT I
# DUE: FEB. $13^{th}$, 2020

(1) Show that a Cauchy sequence in a metric space $(X, d)$ is bounded.

(2) Show that the spaces $l^p, 1 \leq p < \infty$ are complete.
**Hint:** Starting with a Cauchy sequence $\{x^n\}_n, x^n = (x_k^n)_{k=1}^\infty$, show that $x_k := \lim_n x_k^n$ is well-defined. Then, show that $x \in l^p$ and finally $\lim_n \|x^n - x\|_{l^p} = 0$,

(3) Assume that $(X, \|\cdot\|)$ is a normed space over $\mathbb{R}$, whose norm satisfies the parallelogram identity

$$2(\|x\|^2 + \|y\|^2) = \|x+y\|^2 + \|x-y\|^2.$$

Prove that there is a scalar product on $X$, say $p(x, y)$, which in fact generates the norm, i.e. $\|x\| = \sqrt{p(x, x)}$.
**Hint:** Working over $\mathbb{R}$, use the polarization identity to define dot product as follows

$$p(x, y) := \frac{\|x+y\|^2 - \|x-y\|^2}{4}.$$

One issue is to show $p(x_1+x_2, y) = p(x_1, y)+p(x_2, y)$, but I would recommend showing the somewhat more symmetric identity

$$p(x_1 + x_2 + x_3, y) = p(x_1, y) + p(x_2, y) + p(x_3, y). \tag{1}$$

Clearly, (1) implies what you need, with $x_3 = 0$. Once that is done, use to show that $p(\lambda x, y) = \lambda p(x, y)$ for the integers, then rationals and then the reals.

(4) Show that a subspace $K \subset l^2$ is pre-compact if and only
(a) $K$ is bounded
(b) For each $\epsilon > 0$, there exists $N$, so that for each $x = (x_n)_{n=1}^\infty \in K$,

$$\sup_{x \in K} \sum_{n>N} |x_n|^2 < \epsilon^2.$$

**Hint:** For the necessity of $b$), argue by contradiction so that there exists $\epsilon_0 > 0$ and a sequence $x^m \in l^2$, so that $\|x^m_{>m}\|^2 = \left(\sum_{n>m} |x_n^m|^2\right)^{\frac{1}{2}} \geq \epsilon_0$. Take then a convergent subsequence.

For the sufficiency, show that for each sequence $\{x^m\}$ in $K$, the sequences $\{x_j^m\}_j, j = 1, 2, \ldots$ are bounded and hence contain convergent subsequence.

Apply the Cantor diagonalization procedure. Use $b$) to show that the diagonal sequence does converge.

(5) Show that $(l^1)^* = l^\infty$. More precisely, show that $\Lambda : l^1 \to \mathbb{R}$ is a bounded linear functional if and only if there exists unique $y \in l^\infty$, so that

$$\Lambda x = \sum_{n=1}^{\infty} x_n y_n.$$

Moreover $\|\Lambda\|_{(l^1)^*} = \|y\|_{l^\infty}$.

(6) We say that a series $\sum_{j=1}^{\infty} x_j$ in a normed space $(X, \|\cdot\|)$ is absolutely convergent, if $\sum_j \|x_j\| < \infty$. Show that $X$ is complete if and only if every absolutely convergent series is convergent.

**Hint:** In one direction, assuming that every absolutely convergent series is convergent, you need to show every Cauchy sequence is convergent. Consider one $\{a_n\}_n$, and try to extract a subsequence, so that $\|a_{n_{k+1}} - a_{n_k}\| < 2^{-k}$.