01

(1) **(40 points)** Find the general solution of the following second order ODE

$$y''(t) + 4y'(t) + 4y(t) = t^{-2}e^{-2t}.$$

Hint: Use the method of variation of parameters.

$r^2+4r+4=0$

$r_{1,2}=-2$

$$y(t) = u_1(t)e^{-2t} + u_2(t)\cdot t\cdot e^{-2t}$$

$$\rightarrow \begin{cases} u_1'\cdot e^{-2t} + u_2'\cdot t\cdot e^{-2t} = 0 \\ u_1'(-2e^{-2t}) + u_2'(e^{-2t} - 2te^{-2t}) = t^{-2}e^{-2t} \end{cases}$$

$$\rightarrow \begin{cases} u_1' + u_2' t = 0 \\ -2u_1' + u_2'(1-2t) = t^{-2} \end{cases} \rightarrow \begin{cases} u_1' + u_2'\cdot t = 0 \\ -2\underbrace{(u_1' + u_2'\cdot t)}_{=0} + u_2' = t^{-2} \end{cases}$$

$$\rightarrow u_2' = \frac{1}{t^2} \rightarrow u_2 = -\frac{1}{t} + c_2$$

$$u_1' = -t\cdot u_2' = -\frac{1}{t}$$

$$\rightarrow u_1 = -\ln t + c_1$$

$$\rightarrow y(t) = -e^{-2t}\cdot \ln t - e^{-2t} + c_1 e^{-2t} + c_2 t e^{-2t}$$

02

(2) **(40 points)** A 16-pound weight stretches a spring 2 feet. The medium through which the weight moves offers a resistance equal to 4 times the velocity. Find the position of the point for any t, if the weight is released from a point one foot below the equilibrium position with zero velocity. Find the (quasi) amplitude.

$w=16$; $L=2$; $\gamma=4$; $u(0)=1, u'(0)=0$

$$\rightarrow m=\frac{w}{g}=\frac{16}{32}=\frac{1}{2} \; ; \quad K=\frac{w}{L}=8$$

Thus
$$\begin{cases} \frac{1}{2}u''+4u'+8u=0 \\ u(0)=1,\ u'(0)=0 \end{cases}$$

$$\frac{1}{2}r^2+4r+8=0 \rightarrow r^2+8r+16=0$$

$$(r+4)^2=0$$

$$r_{1,2}=-4$$

$$u(t)=C_1e^{-4t}+C_2te^{-4t} \; ; u'=-4C_1e^{-4t}+C_2(e^{-4t}-4te^{-4t})$$

$$1=u(0)=C_1 \; ; \; 0=u'(0)=-4C_1+C_2$$

$$\rightarrow C_2=4$$

$$\boxed{u(t)=e^{-4t}+4te^{-4t}}$$

(3) (**40 points**) A mass weighing 3 lbs stretches a spring 3 in. If the mass is pushed upward, contracting the spring by 1 in and set in motion by a downward velocity of 2 ft/sec and if there is no damping, find the position $u(t)$ at any time t. Determine the frequency, the period and the amplitude of the motion.

$w = 3$ lbs ; $L = 3 \text{ in} = \frac{1}{4}$ ft

$u(0) = -1 \text{ in} = -\frac{1}{12}$ ft ; $u'(0) = 2$

$\gamma = 0$

$\rightarrow m = \frac{w}{g} = \frac{3}{32}$; $K = \frac{w}{L} = \frac{3}{\frac{1}{4}} = 12$

$$\begin{cases} \frac{3}{32} u'' + 12u = 0 \\ u(0) = -\frac{1}{12} ; u'(0) = 2 \end{cases}$$

$\longrightarrow r^2 + 128 = 0$

$r = \pm 8\sqrt{2}\, i$

$u(t) = A \cos(8\sqrt{2}t) + B \sin(8\sqrt{2}t)$;

$-\frac{1}{12} = u(0) = A$; $2 = u'(0) = 8\sqrt{2} B \rightarrow B = \frac{1}{4\sqrt{2}}$

$u(t) = -\frac{1}{12} \cos(8\sqrt{2}t) + \frac{1}{4\sqrt{2}} \sin(8\sqrt{2}t)$

$R = \sqrt{A^2 + B^2} = \sqrt{\frac{1}{12^2} + \frac{1}{32}} = \cdots$; $\omega = 8\sqrt{2}$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{8\sqrt{2}}$.

04

(4) (**40 points**) Compute the Laplace transform of

$$t\sin(at)$$

Start with $\mathcal{L}(\sin(at)) = \frac{1}{s^2+a^2}$

$$\int_0^{+\infty} e^{-st} \sin(at)\,dt = \frac{a}{s^2+a^2}.$$

Take a derivative with respect to s

$$\int_0^{+\infty} e^{-st} \cdot (-t) \cdot \sin(at)\,dt = \frac{-2s.a}{(s^2+a^2)^2}$$

$\longrightarrow \mathcal{L}(t\sin(at)) = \frac{2as}{(s^2+a^2)^2}$

05

(5) **(40 points)** Solve the equation

$$\begin{cases} y'' + y = u_{3\pi}(t) \\ y(0) = 0, y'(0) = 1. \end{cases}$$

$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(u_{3\pi})$; $F(s) = \mathcal{L}(y)$

$(s^2 F(s) - 1) + F(s) = \dfrac{e^{-3\pi s}}{s}$

$\rightarrow F(s) = \dfrac{1}{1+s^2} + \dfrac{e^{-3\pi s}}{s(s^2+1)}$

Now $G(s) = \dfrac{1}{s(s^2+1)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+1} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$

$\mathcal{L}^{-1}(G) = 1 - \cos(t)$

$\rightarrow y(t) = \mathcal{L}^{-1}(F) = \sin t + u_{3\pi}(t).(1 - \cos(t - 3\pi))$

$$\boxed{y(t) = \sin t + u_{3\pi}(t).(1 + \cos t)}$$