P1

(1) **(30 points)** Find the general solution of the following second order ODE

$$y''(t) + 4y'(t) + 4y(t) = t^{-3}e^{-2t}.$$

Hint: Use the method of variation of parameters.

$(r+2)^2=0$

$y(t) = u_1(t)e^{-2t} + u_2(t) \cdot te^{-2t}$

$\rightarrow \begin{cases} u_1' e^{-2t} + u_2' \cdot t \cdot e^{-2t} = 0 \\ u_1'(-2e^{-2t}) + u_2'(-2t \cdot e^{-2t} + e^{-2t}) = t^{-3}e^{-2t} \end{cases}$

$\rightarrow \begin{cases} u_1' + tu_2' = 0 \\ -2\underbrace{(u_1' + tu_2')}_{=0} + u_2' = t^{-3} \end{cases} \longrightarrow u_2' = t^{-3} \rightarrow u_2 = -\frac{t^{-2}}{2} + C_2$

$u_1' = -tu_2' = -t^{-2}$

$u_1 = \frac{1}{t} + C_1$

$\rightarrow y(t) = \frac{e^{-2t}}{t} + \left(-\frac{1}{2t^2}\right) \cdot te^{-2t} + C_1 e^{-2t} + C_2 te^{-2t}$

$= \frac{1}{2}\frac{e^{-2t}}{t} + C_1 e^{-2t} + C_2 te^{-2t}$

O2

(2) **(40 points)** A 16-pound weight stretches a spring 2 feet. The medium through which the weight moves offers a resistance equal to 4 times the velocity. Find the position of the point for any t, if the weight is released from a point one foot below the equilibrium position with zero velocity. Find the (quasi) amplitude.

$w=16$; $L=2$; $\gamma=4$; $u(0)=1, u'(0)=0$

$$\rightarrow m=\frac{w}{g}=\frac{16}{32}=\frac{1}{2} \quad ; \quad k=\frac{w}{L}=8$$

Thus
$$\begin{cases} \frac{1}{2}u''+4u'+8u=0 \\ u(0)=1,\ u'(0)=0 \end{cases}$$

$$\frac{1}{2}r^2+4r+8=0 \rightarrow r^2+8r+16=0$$

$$(r+4)^2=0$$

$$r_{1,2}=-4$$

$$u(t)=C_1e^{-4t}+C_2te^{-4t} \ ; u'=-4C_1e^{-4t}+C_2(e^{-4t}-4te^{-4t})$$

$$1=u(0)=C_1 \quad ; \quad 0=u'(0)=-4C_1+C_2$$

$$\rightarrow C_2=4$$

$$\boxed{u(t)=e^{-4t}+4te^{-4t}}$$

(3) (**40 points**) A mass weighing 3 lbs stretches a spring 3 in. If the mass is pushed upward, contracting the spring by 1 in and set in motion by a downward velocity of 2 ft/sec and if there is no damping, find the position $u(t)$ at any time t. Determine the frequency, the period and the amplitude of the motion.

$w = 3$ lbs ; $L = 3$ in $= \frac{1}{4}$ ft

$u(0) = -1$ in $= -\frac{1}{12}$ ft ; $u'(0) = 2$

$\gamma = 0$

$$\rightarrow m = \frac{w}{g} = \frac{3}{32} \ ; \ K = \frac{w}{L} = \frac{3}{\frac{1}{4}} = 12$$

$$\begin{cases} \frac{3}{32} u'' + 12u = 0 \\ u(0) = -\frac{1}{12} \ ; \ u'(0) = 2 \end{cases} \longrightarrow r^2 + 128 = 0$$

$$r = \pm 8\sqrt{2}\, i$$

$$u(t) = A\cos(8\sqrt{2}t) + B\sin(8\sqrt{2}t) \ ;$$

$$-\frac{1}{12} = u(0) = A \ ; \ 2 = u'(0) = 8\sqrt{2}\, B \rightarrow B = \frac{1}{4\sqrt{2}}$$

$$u(t) = -\frac{1}{12}\cos(8\sqrt{2}t) + \frac{1}{4\sqrt{2}}\sin(8\sqrt{2}t)$$

$$R = \sqrt{A^2 + B^2} = \sqrt{\frac{1}{12^2} + \frac{1}{32}} = \cdots \ ; \ \omega = 8\sqrt{2}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{8\sqrt{2}}.$$

(4) (**40 points**) Compute the Laplace transform of

$$t\cos(at)$$

Start with $\mathcal{L}(\cos at) = \dfrac{s}{s^2+a^2}$

$$\int_0^{+\infty} e^{-st}\cos(at)\,dt = \frac{s}{s^2+a^2}$$

Take a derivative with respect to s:

$$\int_0^{+\infty} e^{-st}(-t)\cos(at)\,dt = \frac{(s^2+a^2)-2s\cdot s}{(s^2+a^2)^2} = \frac{a^2-s^2}{(s^2+a^2)^2}$$

$$\rightarrow \mathcal{L}(t\cdot\cos(at)) = \frac{s^2-a^2}{(a^2+s^2)^2}$$

p5

(5) **(40 points)** Solve the equation

$$\begin{cases} y'' + y = u_{3\pi}(t) \\ y(0) = 1, y'(0) = 0. \end{cases}$$

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(u_{3\pi}) \quad ; \quad F(s) = \mathcal{L}(y)(s)$$

$$(s^2 F(s) - s) + F(s) = \frac{e^{-3\pi s}}{s}$$

$$\longrightarrow \quad F(s) = \frac{s}{s^2+1} + \frac{e^{-3\pi s}}{s(s^2+1)}$$

$$G(s) = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$g(t) = \mathcal{L}^{-1}(G(s))(t) = 1 - \cos(t)$$

Thus

$$y(t) = \cos t + u_{3\pi}(t)\left(1 - \cos(t - 3\pi)\right)$$

$$y(t) = \cos t + u_{3\pi}(t)\left(1 + \cos t\right)$$