(1) **(30 points)** Solve the initial value problem:

$$\begin{vmatrix} ty' + 2y = t^2 + 1, \quad t > 0 \\ y(1) = 1, \end{vmatrix}$$

$$y' + \frac{2}{t}y = \frac{t^2+1}{t} \longrightarrow \mu(t) = e^{\int \frac{2}{t}} = e^{2\ln t} = t^2$$

$$\rightarrow (t^2 y)' = t(t^2+1)$$

$$t^2 y(t) = \int t(t^2+1)dt = \frac{(t^2+1)^2}{4} + C$$

$$y = \frac{(t^2+1)^2}{4t^2} + \frac{C}{t^2} \;; \quad 1 = y(1) = 1 + C \rightarrow C = 0$$

$$y = \frac{(t^2+1)^2}{4t^2}$$

(2) **(30 points)** Find the general solution of

$$y' = \frac{x^2y^2}{\sqrt{x^3+1}}.$$

$$\frac{y'}{y^2} = \frac{x^2}{\sqrt{x^3+1}} \longrightarrow \int \frac{dy}{y^2} = \int \frac{x^2}{\sqrt{x^3+1}}\,dx$$

$$\rightarrow -\frac{1}{y} = \frac{2}{3}\sqrt{x^3+1} + C$$

$$y = -\frac{1}{C + \frac{2}{3}\sqrt{x^3+1}}$$

(3) (**30 points**) Find the general solution of

$$\left(\frac{y}{x}+6x\right)dx+(\ln(x)-2)dy=0$$

$M=\frac{y}{x}+6x\ ;\ N=\ln x-2\ ;\ M_y=\frac{1}{x}=N_x$

$$\begin{cases}\psi_x = M = \frac{y}{x}+6x \\ \psi_y = N = \ln(x)-2\end{cases} \rightarrow \quad \psi=\int \psi_y\, dy = y\ln x - 2y + C(x)$$

$\rightarrow \cancel{\frac{y}{x}}+6x=\psi_x=\cancel{\frac{y}{x}}+C'(x)$

$\rightarrow C(x)=3x^2$

$\rightarrow C=\psi(x,y)=y\ln x-2y+3x^2$

(4) (**30 points**) Solve the initial value problem $4y''-y=0, y(0)=a, y'(0)=1$.
Find a so that the solution approaches zero as $t\rightarrow\infty$.

$4r^2-1=0 \rightarrow r_{1,2}=\pm\frac{1}{2}$

$y=C_1e^{t/2}+C_2e^{-t/2}$

$y'=\frac{1}{2}C_1e^{t/2}-C_2e^{-t/2}$

$$\rightarrow \begin{cases} a=y(0)=C_1+C_2 \\ 1=y'(0)=\frac{1}{2}(C_1-C_2)\end{cases} \rightarrow C_1=\frac{a+2}{2},\ C_2=\frac{a-2}{2}$$

$\rightarrow y=\frac{a+2}{2}e^{t/2}+\frac{a-2}{2}.e^{-t/2}$.

The solution $y(t)\rightarrow 0$, only if $\frac{a+2}{2}=0$

or $\boxed{a=-2}$

(5) **(40 points)** Solve

$$\left| \begin{array}{l} y''(t) + 4y(t) = \sin(2t) \\ y(0) = 0, y'(0) = 0. \end{array} \right.$$

$$r^2 + 4 = 0 \longrightarrow r = \pm 2i$$

$$y_{homog.}(t) = C_1 \cos(2t) + C_2 \sin(2t)$$

Thus, right-hand side is <u>resonant.</u>

$$Y_0(t) = t\left(A\sin(2t) + B\cos(2t)\right)$$

$$Y_0' = A\sin(2t) + B\cos(2t) + t\left(2A\cos(2t) - 2B\sin(2t)\right)$$

$$Y_0'' = +2A\cos(2t) - 2B\sin(2t) + \left(2A\cos(2t) - 2B\sin(2t)\right)$$

$$+ t\left(-4A\sin(2t) - 4B\cos(2t)\right)$$

$$\rightarrow \left(\cancel{-4At\sin(2t)} - 4Bt\cancel{\cos(2t)}\right) + 4A\cos(2t) - 4B\sin(2t)$$

$$+ 4t\left(A\cancel{\sin(2t)} + B\cancel{\cos(2t)}\right) = \sin(2t)$$

$$\longrightarrow \begin{cases} 4A = 0 \\ -4B = 1 \end{cases} \longrightarrow \begin{array}{l} A = 0 \\ B = -\frac{1}{4} \end{array} \rightarrow Y_0 = -\frac{1}{4} t\cos(2t)$$

$$\rightarrow y(t) = C_1 \cos(2t) + C_2\sin(2t) - \frac{1}{4} t\cos(2t)$$

$$0 = y(0) = C_1 \rightarrow C_1 = 0$$

$$0 = y'(0) = 2C_2 - \frac{1}{4} \rightarrow C_2 = \frac{1}{8}$$

$$\rightarrow y(t) = \frac{1}{8}\sin(2t) - \frac{1}{4} t\cos(2t)$$

(6) (**40 points**) A tank with a capacity of 600 gal originally contains 200 gal of water with 100 lb of salt in it. Water containning 1 lb of salt per galon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when it begins to overflow. Find the concentration (in *lbs/gal.*) at the time of overflowing.

$$\begin{cases} Q'(t) = 3 - \dfrac{Q(t)}{200+t} \cdot 2 \\ Q(0) = 100 \end{cases} \qquad 0 \leq t \leq 400$$

$$Q' + \frac{2}{200+t} Q = 3 \quad , \mu = e^{2\int \frac{dt}{200+t}} = e^{2\ln(200+t)} = (200+t)^2$$

$$\rightarrow \left(Q(t)(200+t)^2\right)' = 3(200+t)^2$$

$$Q(t)(200+t)^2 = 3\int (200+t)^2 dt = (200+t)^3 + C$$

$$Q(t) = (200+t) + \frac{C}{(200+t)^2}$$

$$100 = Q(0) = 200 + \frac{C}{200^2} \rightarrow C = -100(200)^2$$

$$\rightarrow Q(t) = (200+t) - \frac{100(200)^2}{(200+t)^2}$$

$$Q(400) = 600 - \frac{100(200)^2}{(600)^2} = 600 - \frac{400}{36} \approx 589 \text{ lbs.}$$

$$\text{Concentration} = \frac{Q(400)}{600} = \frac{589 \text{ lbs}}{600 \text{ gal}} \approx 1 \text{ lb/gal.}$$