(1) **(30 points)** Solve the initial value problem:

$$\left| \begin{array}{l} y' + 2ty = 2te^{-t^2}, \quad t > 0 \\ y(0) = 1, \end{array} \right.$$

$$\mu = e^{\int 2t\,dt} = e^{t^2} \rightarrow \left(e^{t^2} y(t)\right)' = 2t$$

$$e^{t^2} y(t) = \int 2t\,dt = t^2 + C$$

$$\rightarrow \quad y(t) = t^2 e^{-t^2} + Ce^{-t^2}; \quad 1 = y(0) = C$$

$$\rightarrow y(t) = (t^2+1)e^{-t^2}$$

(2) **(30 points)** Find the general solution of

$$y' = \frac{xy^3}{\sqrt{x^2+1}}.$$

$$\frac{y'}{y^3} = \frac{x}{\sqrt{x^2+1}} \rightarrow \int \frac{dy}{y^3} = \int \frac{x\,dx}{\sqrt{x^2+1}}$$

$$-\frac{1}{2y^2} = \sqrt{(x^2+1)} + C$$

$$\rightarrow \boxed{\frac{1}{2y^2} + \sqrt{x^2+1} = C}$$

(3) (**30 points**) Find the general solution of

$$\underbrace{(\ln(y)-2)}_{M}dx + \underbrace{\left(\frac{x}{y}+6y\right)}_{N}dy = 0$$

$M_y = \frac{1}{y} = N_x \longrightarrow \begin{cases} \psi_x = \ln y - 2 \\ \psi_y = \frac{x}{y} + 6y \end{cases}$

$\rightarrow \psi = \int \psi_x dx = \int (\ln y - 2) dx = x \ln y - 2x + C(y)$

$\cancel{\frac{x}{y}} + 6y = \psi_y = \cancel{\frac{x}{y}} + C'(y) \rightarrow C'(y) = 6y$

$\rightarrow C(y) = 3y^2$

Solution is: $\quad x \ln y - 2x + 3y^2 = C$

(4) (**30 points**) Solve the initial value problem $4y'' - y = 0, y(0) = 2, y'(0) = a$. Find a so that the solution approaches zero as $t \to \infty$.

$4r^2 - 1 = 0 \longrightarrow r_{1,2} = \pm \frac{1}{2}$

$y(t) = C_1 e^{t/2} + C_2 e^{-t/2}$; $y'(t) = \frac{1}{2} C_1 e^{t/2} - \frac{C_2}{2} e^{-t/2}$

$\begin{cases} 2 = C_1 + C_2 \\ a = \frac{1}{2}(C_1 - C_2) \end{cases} \longrightarrow \begin{array}{l} C_1 = a+1 \\ C_2 = 1-a \end{array}$

$\rightarrow y(t) = (a+1) e^{t/2} + (1-a) e^{-t/2}$.

$y(t) \rightarrow 0$ only if $(a+1) = 0 \longrightarrow \boxed{a = -1}$

(5) **(40 points)** Solve

$$\left| \begin{array}{l} y''(t) + 4y(t) = 4\cos(2t) \\ y(0) = 0, y'(0) = 1 \end{array} \right.$$

$r^2+4=0 \longrightarrow r_{1,2} = \pm 2i$

$y_{homg.} = C_1 \cos(2t) + C_2 \sin(2t)$

Thus, we have ~~a~~ resonance.

$Y_0(t) = t(A\cos(2t) + B\sin(2t))$

$Y_0' = A\cos(2t) + B\sin(2t) + t(-2A\sin(2t) + 2B\cos(2t))$

$Y_0'' = -2A\sin(2t) + 2B\cos(2t) - 2A\sin(2t) + 2B\cos(2t)$
$+ t(-4A\cos(2t) - 4B\sin(2t))$

$(\cancel{-4At\cos(2t)} \cancel{-4Bt\sin(2t)} - 4A\sin(2t) + 4B\cos(2t))$

$+ \cancel{4t(A\cos(2t) + B\sin(2t))} = 4\cos(2t)$

Thus, $\begin{cases} -4A = 0 \\ 4B = 4 \end{cases} \longrightarrow A=0, B=1, \; Y_0 = t\sin(2t)$

$y(t) = C_1 \cos(2t) + C_2 \sin(2t) + t\sin(2t)$

$0 = y(0) = C_1$; $1 = y'(0) = 2C_2 \longrightarrow C_2 = \frac{1}{2}$

$y(t) = \frac{1}{2}\sin(2t) + t\sin(2t)$

(6) (**40 points**) A tank with a capacity of 600 gal originally contains 200 gal of water with 100 lb of salt in it. Water containning 1 lb of salt per galon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when it begins to overflow. Find the concentration (in *lbs/gal.*) at the time of overflowing.

$$\begin{cases} Q'(t) = 3 - \dfrac{Q(t)}{200+t} \cdot 2 \\ Q(0) = 100 \end{cases} \qquad 0 \le t \le 400$$

$$Q' + \frac{2}{200+t} Q = 3 \quad , \mu = e^{2\int \frac{dt}{200+t}} = e^{2\ln(200+t)} = (200+t)^2$$

$$\rightarrow \left(Q(t)(200+t)^2\right)' = 3(200+t)^2$$

$$Q(t)(200+t)^2 = 3\int (200+t)^2 dt = (200+t)^3 + C$$

$$Q(t) = (200+t) + \frac{C}{(200+t)^2}$$

$$100 = Q(0) = 200 + \frac{C}{200^2} \rightarrow C = -100(200)^2$$

$$\rightarrow Q(t) = (200+t) - \frac{100(200)^2}{(200+t)^2}$$

$$Q(400) = 600 - \frac{100(200)^2}{(600)^2} = 600 - \frac{400}{36} \approx 589 \text{ lbs.}$$

$$\text{Concentration} = \frac{Q(400)}{600} = \frac{589 \text{ lbs}}{600 \text{ gal}} \approx 1 \text{ lb/gal.}$$