Notes MATH 960

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1 Chapter I

1.1 Metric and Banach spaces

We start with a few definitions

Definition 1. *We say that* (X,d) *is a metric space, if* $d: X\times X\to \mathbf{R}_+$, *with the following properties*

1. $d(x,y)\geq 0$ *and* $d(x,y)=0$ *if and only if* $x=y$
2. $d(x,y)=d(y,x)$
3. *(triangle inequality)* $d(x,z)\leq d(x,y)+d(y,z)$.

We say that $U\subset X$ *is an open set, if for every* $x\in U$, *there is an* $\epsilon>0$, *so that* $x\in B_\epsilon(x)=\{y\in X: d y,x)<\epsilon\}\subset U$.

The collection $\tau=\{U\subset X: U-\text{open}\}$ *is called a topology on* X.

Definition 2. *We say that the sequence* $\{x_n\}_n\subset (X,d)$ *is a Cauchy sequence, if for every* $\epsilon>0$, *there is* N, *so that whenever* $n>m>N$, *there is* $d(x_n,x_m)<\epsilon$.
Note: *Every convergent sequence is Cauchy.*

We say that the metric space (X,d) *is complete, if every Cauchy sequence is convergent.*
Note: *In order to show that a Cauchy sequence is convergent, it suffices to find a convergent subsequence.*

Definition 3. *Let X be a vector space, i.e. the operations $x+y$ and ax (where x, y are vectors and a is a scalar) a are well-defined. A function $\|\cdot\| : X \to \mathbf{R}_+$ is called a norm on X, if*

1. $\|x\| \geq 0$ *and* $\|x\| = 0$ *if and only if* $x = 0$
2. $\|ax\| = |a|\|x\|$
3. *(triangle inequality)* $\|x+y\| \leq \|x\| + \|y\|$.

A vector space with a norm $(X, \|\cdot\|)$ is called a normed space. A normed space $(X, \|\cdot\|)$, which is complete in the metric $d(x, y) = \|x - y\|$ is called a Banach space.

A few examples:

- $l^p, 1 \leq p < \infty$,
$$l^p = \left\{ x = (x_n)_{n=1}^{\infty} : \|x\| := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \right\}.$$
- For a measure space $(M, d\mu)$, $L^p(M, d\mu), 1 \leq p < \infty$,
$$L^p(M, d\mu) = \left\{ f : M \to \mathbf{R} : \|f\| := \left(\int_M |f(x)|^p d\mu \right)^{\frac{1}{p}} \right\}.$$
- $L^\infty(M, d\mu)$,
$$L^\infty(M, d\mu) = \left\{ f : M \to \mathbf{R} : \|f\| := esssup\{|f(x)| : x \in M\right\}.$$

1.2 Compactness

Definition 4. *We say that $K \subset (X, d)$ is compact, if every open cover $K \subset \cup_\alpha U_\alpha$ has a finite subcover, i.e. there exists $\alpha_1, \dots, \alpha_N$, so that $K \subset \cup_{j=1}^N U_{\alpha_j}$.*

We say that $K \subset (X, d)$ is sequentially compact, if every sequence $\{x_n\}$ in K has a convergent subsequence converging to $x \in K$. We say that K is precompact, if $\overline{K}$ is compact.

Proposition 1. *Let (X,d) be a complete metric space. Then, $K \subset (X,d)$ is compact if and only if K is sequentially compact.*

Equivalently, $K \subset (X,d)$ is pre-compact if and only if every sequence $\{x_n\}$ in K has a convergent subsequence.

In specific cases, one can characterize compactness efficiently.

1.2.1 Compacts in $\mathscr{C}(X,\mathbb{C})$

Let (X,d) be a compact metric space. Introduce the space of continuous functions on X

$$\mathscr{C}(X,\mathbb{C}) = \{f : X \to \mathbb{C} : f \text{ is continuous}, \|f\| = \sup_{x\in K} |f(x)|\}.$$

Theorem 1. *(Arzela-Ascolli)*

The subset $K \subset \mathscr{C}(X,\mathbb{C})$ is pre compact if and only if

1. *K is bounded, i.e.*
$$\sup_{f\in K} \|f\| < \infty.$$
2. *K is equi-continuous. That is, for every $\epsilon > 0$, there exists $\delta > 0$, so that for each $x, x' \in X : d(x,x') < \delta$,*
$$\sup_{f\in K} |f(x) - f(x')| < \epsilon.$$

1.2.2 Compacts in c_0, l^p spaces

Theorem 2. *$K \subset l^p, 1 \le p < \infty$ is pre-compact if and only if*

1. *K is bounded, i.e.*
$$\sup_{x\in K} \|x\|_{l^p} < \infty.$$
2. *For every $\epsilon > 0$, there exists N, so that*
$$\sup_{x\in K} \sum_{n=N}^{\infty} |x_n|^p < \epsilon.$$

$K \subset c_0, 1$ *is pre-compact if and only if*

1. *K is bounded, i.e.*
$$\sup_{x \in K} \|x\|_{c_0} < \infty.$$
2. *For every $\epsilon > 0$, there exists N, so that*
$$\sup_{x \in K} \sup_{n \geq N} |x_n| < \epsilon.$$

1.3 Bounded linear operators

Definition 5. *Let $(X, \|\cdot\|), (Y, \|\cdot\|)$ be normed spaces. A bounded linear operator $A : X \to Y$ is said to be bounded, if $\|Ax\|_Y \leq C\|x\|_X$.*

Proposition 2. *The space $B(X, Y) = \{A : X \to Y; A-\text{ bounded linear operator}\}$ can be made a normed space via the norm*
$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|}.$$

Note the inequality $\|Ax\| \leq \|A\|\|x\|$.

1.3.1 Finite dimensional spaces

We say that X is finite dimensional, if $X = span[e_1, e_2, \ldots, e_n]$ and $\{e_j\}_{j=1}^n$ is linearly independent. In such case $dim(X) := n$.

Theorem 3. *All norms on X are equivalent. That is, for any norm $\|x\|$ on X, there is a $C > 1$, so that $C^{-1}\|x\| \leq \|x\|_1 \leq C\|x\|$, where $\|\sum_{j=1}^n \lambda_j e_j\|_1 := \sum_{j=1}^n |\lambda_j|$.*

In other words, X is isomorphic to $\mathbf{R}^n$ or $\mathbf{C}^n$. Thus, the compactness is the same, so the Heine-Borel theorem holds

Corollary 1. *$K \subset X$, with $dim(X) = n$ is compact if and only if*

1. *K is bounded*

2. *K is closed.*

The next theorem states that this previous result characterizes finite dimensional spaces.

Theorem 4. *Let $(X, \|\cdot\|)$ be a normed space. Then, the following are equivalent*

1. $dim(X) < \infty$
2. $B_X = \{x \in X : \|x\| \leq 1\}$ *is compact.*
3. $S_X = \{x \in X : \|x\| = 1\}$ *is compact.*

1.3.2 Quotient spaces

Let $(X, \|\cdot\|)$ be a normed space, $Y \subset X$, Y is a closed subspace. Then, define equivalence relation $x_1 \sim x_2 : x_1 - x_2 \in Y$. This introduces equivalence classes $[x] = \{\tilde{x} \in X : \tilde{x} \sim x\}$.

Lemma 1. *The quotient space $X/Y = \{[x] : x \in X\}$ is a normed space, under the norm*

$$\|[x]\| = \inf_{y \in Y} \|x + y\|.$$

Moreover, if X is a Banach space, then X/Y is Banach space as well.

1.4 Dual spaces

Definition 6. *For a normed space $(X, \|\cdot\|)$, we say that its dual space is $X^* = L(X, \mathbf{R})$. That is, X^* is the space of continuous linear functionals on X.*

Examples:

- Hilbert space H, with a dot product $\langle x, y \rangle$. Its dual space can be identified with itself[1].

[1] This is in essence the Riesz representation theorem, see below

- $L^p(M, d\mu), 1 \le p < \infty$. The dual space is $(L^p(M, d\mu))^* = L^q(M, d\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, every $\Lambda \in (L^p(M, d\mu))^*$ has the form

$$\Lambda f = \int_M f g d\mu, g \in L^q(M, d\mu), \|\Lambda\|_{(L^p)^*} = \|g\|_{L^q}$$

  In particular, $(l^p)^* = l^q, 1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

- $c_0 = \{x = (x_n)_{n=1}^\infty : \lim_n x_n = 0, \|x\|_{c_0} = \sup_n |x_n|\}$. Its dual is $c_0^* = l^1$.

- $(l^1)^* = l^\infty$. Note however that $(l^\infty)^* \supsetneq l^1$.

1.5 Hilbert spaces

Definition 7. *Let H be a real vector space. If a bilinear form $\langle \cdot, \cdot \rangle : H \times H \to \mathbf{R}$ has the properties*

- $\langle x, x \rangle \ge 0$ *and* $\langle x, x \rangle = 0$, *if and only if* $x = 0$

- $\langle x, y \rangle = \langle y, x \rangle$

then we say that $\langle \cdot, \cdot \rangle$ is a dot product on H.

In this case define $\|x\| := \sqrt{\langle x, x \rangle}$. In order to check it is defining a norm on H, we need the following lemma.

Lemma 2.
- $|\langle x, y \rangle| \le \|x\| \|y\|$ *(Cauchy-Schwartz inequality)*

- $\|x + y\| \le \|x\| + \|y\|$.

Note that the Hilbert space norm satisfies the *parallelogram identity*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Theorem 5. *(Riesz representation theorem)*

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Then for any bounded linear functional Λ on H, there exists an unique $y \in H$, so that $\Lambda x = \langle x, y \rangle$. Moreover, $\|\Lambda\|_{H^*} = \|y\|$. In other words, $H^* = H$.

Definition 8. *Let $S \subset H$, H - Hilbert space.* **Orthogonal complement** *is*

$$S^{\perp} = \{x \in H : \langle x, y \rangle = 0 \forall y \in S\}.$$

Note that $S^{\perp}$ is always a closed subspace of H.

Definition 9. *We say that a Banach space X is a direct sum of two subspaces $X = Y_1 \oplus Y_2$, if $Y_1 \cap Y_2 = \emptyset$ and $X \subset Y_1 + Y_2 = \{y_1 + y_2 : y_j \in Y_j, j = 1, 2\}$.*
Note that in such case, for every $x \in X$, there is unique pair $y_1 \in Y_1, y_2 \in Y_2$, so that $x = y_1 + y_2$.

Proposition 3. *Let H be a Hilbert space and $E \subset H$ is a subspace of it. Then*

$$H = E \oplus E^{\perp}.$$

1.6 Baire category theorem

Definition 10. *Let (X, d) be a complete metric space. Then,*

- *A is called* **nowhere dense**, *if $Int(\bar{A}) = \emptyset$.*
- *A is called* **meager**, *if $A \subset \cup_{j=1}^{\infty} A_j$, where A_j are nowhere dense.*
- *Ω is called* **residual**, *if Ω^c is meager.*

Note that if A is meager, then $\bar{A}$ is also meager. Ω is residual, if $\Omega \supset \cap_{j=1}^{\infty} U_j$, where U_j are open and dense. These are called G_δ sets.

Theorem 6. *(Baire category theorem)*

Let (X, d) be a metric space. Let $\{U_j\}_{j=1}^{\infty}$ be a family of open and dense sets. Then $\cap_{j=1}^{\infty} U_j$ is a dense set in X.

Corollary 2. *Let (X, d) be a metric space. Then,*

- *If Ω is residual, then Ω is dense.*
- *$X \neq \cup_{j=1}^{\infty} F_j$, where F_j is nowhere dense. Moreover, $Int(\cup_{j=1}^{\infty} F_j) = \emptyset$.*

The way this is used in practice is as follows: If $X = \cup_{j=1}^{\infty} F_j$ and F_j are closed, then there exists j_0, so that $Int(F_{j_0}) \neq \emptyset$.

2 Chapter II

2.1 Uniform boundedness principle

Theorem 7. *(UBP)*

Let $(X, \|\cdot\|)$ *and* $(Y_i, \|\cdot\|_i)$ *are Banach spaces,* $i \in I$*. Suppose that* $A_i : X \to Y_i$ *are bounded linear operators. Suppose that for every* $x \in X$*, the orbit* $\{A_i x\}$ *is bounded. That is* $\sup_i \|A_i x\| < \infty$*. Then,*

$$\sup_i \|A_i\| < \infty.$$

Corollary 3. *Suppose that* $(X, \|\cdot\|)$ *and* $(Y_i, \|\cdot\|_i), i \in I$ *are Banach spaces. Suppose that* $\sup_i \|A_i\| = \infty$*. Then, there exists* $x \in X$*, so that* $\sup_{i \in I} \|A_i x\| = \infty$*.*

Another result, which is very useful in the application is the Banach-Steinhaus theorem.

Theorem 8. *(Banach-Steinhaus)*

Let X, Y *are Banach spaces and* $A_n : X \to Y$ *is a sequence of bounded linear operators, so that* $\lim_n A_n x$ *exists for every* x*. Then*

- $\sup_n \|A_n\| < \infty$
- $AX := \lim_n A_n x$ *is a bounded linear operator*

2.2 Open mapping theorem

Definition 11. *We say that a mapping* $f : X \to Y$ *is open, if for every open set* $U \subset X$*,* $f(U)$ *is open.*

Theorem 9. *(Open mapping theorem)*

Let X, Y *are Banach spaces and* $T : X \to Y$ *is a bounded linear operator, which is onto, i.e.* $T(X) = Y$*. Then,* T *is an open mapping.*

One of the main applications is the inverse operator theorem.

Theorem 10. *Let X, Y are Banach spaces and $T : X \to Y$ is a bounded linear operator, which is a bijection, i.e. one-to-one and onto Then, the algebraically defined inverse operator $T^{-1} : Y \to X$ is bounded.*

Some corollaries are as follows.

Corollary 4. *Suppose that X is a Banach space, so that $X = X_1 \oplus X_2$. Then, there exists a constant c, so that for every $x = x_1 + x_2$,*

$$\|x_1\| + \|x_2\| \leq c\|x_1 + x_2\|.$$

2.3 Hahn-Banach theorem

Definition 12. *Let X be a real vector space. We say that $p : X \to \mathbf{R}$ is a quasi semi-norm, if*

1. *For each $\lambda > 0$, $p(\lambda x) = \lambda p(x)$*
2. *$p(x + y) \leq p(x) + p(y)$.*

If we have $p(\lambda x) = |\lambda| p(x)$ for each $\lambda \in \mathbf{R}$, we say that p is a semi-norm.

Examples:

1. $p(x) = \|x\|$
2. For a convex set $K \subset X$, define its Minkowski functional

$$p_K(x) = \inf\{a > 0 : \frac{x}{a} \in K\}.$$

3. On l^∞, $p(x) = \limsup_n x_n$.

Theorem 11. *(Hahn-Banach theorem)*

Let X be a real normed space and $p : X \to \mathbf{R}$ be a quasi semi-norm on it. Let $Y \subset X$ be a linear subspace and $\phi : Y \to \mathbf{R}$ is a linear functional, so that $\phi(y) \leq p(y)$. Then, there exists an extension $\Phi : X \to \mathbf{R}$, so that $\Phi|_Y = \phi$ and $\Phi(x) \leq p(x), x \in X$.

Corollary 5. *(Complex version)*

Let X be a complex Banach space and $Y \subset X$ be a linear subspace. Let $\psi : Y \to \mathbb{C}$ be a complex linear functional, so that $|\psi(x)| \le c\|x\|$. Then, there exists an extension $\Psi : X \to \mathbb{C}$, so that $\Psi|_Y = \psi$ and $|\Psi(x)| \le c\|x\|, x \in X$.

Lemma 3. *(Properties of the Minkowksi functional)*

Let X be real topological vector space and $K \subset X$ is a convex set, with $0 \in Int(K)$. Then, its Minkowksi functional

$$p_K(x) = \inf\{a > 0 : \frac{x}{a} \in K\}$$

satisfies $p(\lambda x) = \lambda p(x)$ for each $\lambda > 0$, $p_K(x+y) \le p_K(x) + p_K(y)$.

2.4 Applications of Hahn-Banach theorem

2.4.1 Separation of convex sets

Theorem 12. *(Hyperplane separation theorem) Let X be a real topological vector space and $K \subset X$ be convex and open subset. Then, for every $y \notin K$, there exists a linear functional $\Lambda : X \to \mathbf{R}$ and $c \in \mathbf{R}$, so that*

$$\sup_{x \in K} \Lambda(x) \le c = \Lambda(y).$$

Theorem 13. *(Separation of two convex sets)*

Let X be a real topological vector space and $A, B \subset X$ are convex subsets, so that $Int(A) \neq \emptyset$ and $A \cap B = \emptyset$. Then, there exists a linear functional $\Lambda : X \to \mathbf{R}$ and $c \in \mathbf{R}$, so that

$$\sup_{x \in A} \Lambda(x) \le c \le \inf_{y \in B} \Lambda(y)$$

2.4.2 Closure of a linear subspace

For any normed space X and its dual X^*, introduce the notation

$$\langle x, x^* \rangle := x^*(x),$$

to denote the action of x^* on x. Clearly, this allows us to view $x \in X \subset X^{**}$, via the formula $x(x^*) = \langle x, x^* \rangle$.

Definition 13. *For any X real normed vector space and any set $S \subset X$, define its annihilator*

$$S^{\perp} = \{x^* \in X^* : \langle s, x^* \rangle = 0, \ \ \forall s \in S\}.$$

Note: *$S^{\perp}$ is a always a closed linear subspace of X^*.*

Theorem 14. *Let X be a Banach space, $Y \subset X$ be a linear subspace, so that $x_0 \in X \setminus \bar{Y}$. Then, $\delta = dist(x_0, Y) > 0$ and there exists $x^* \in Y^{\perp}$, so that $\|x^*\| = 1$, $x^*(x_0) = \delta$.*

Corollary 6. *For every $x \in X$, there is an $x^* \in X^* : \|x^*\| = 1, x^*(x) = \|x\|$.*

Corollary 7. *Let X be a Banach space, $Y \subset X$ be a linear subspace,Then,*

$$x \in \bar{Y} \iff \langle x, x^* \rangle = 0, \forall x^* \in Y^{\perp}.$$

One might introduce for every $S \subset X^*$, $S^{\top} = \{x \in X : \langle x, s \rangle = 0, \ \ \forall s \in S\}$. In this case, Corollary 7 reads

$$\bar{Y} = (Y^{\perp})^{\top}.$$

3 Weak and Weak$*$ topology

Let X be a real vector space and $\mathscr{F}$ be a collection of real-valued linear functionals. Define the sets

$$\mathcal{V}_{\mathscr{F}} = \{\cap_{i=1}^{m} f_i^{-1}(a_i, b_i) : f_i \in \mathscr{F}, a_i < b_i\}.$$

Lemma 4. *The set*

$$\mathcal{U}_{\mathscr{F}} = \{U \subset X : \forall x \in U, \exists V \in \mathcal{V}_{\mathscr{F}}, x \in V \subseteq U\}.$$

is a topology on X. In other words, $\mathcal{V}_{\mathscr{F}}$ is a base for $\mathcal{U}_{\mathscr{F}}$, while the sets $\{f^{-1}(a,b) : f \in \mathscr{F}, a < b\}$ are a sub-base for $\mathcal{U}_{\mathscr{F}}$.

The topology $(X, \mathcal{U}_{\mathscr{F}})$ may be equivalently defined by saying that $x_\alpha \to x$ exactly when $f(x_\alpha) \to f(x)$ for each $f \in \mathscr{F}$.

3.1 Weak topology on X

Definition 14. *(Weak topology on X)*

Let X real normed vector space and $\mathcal{F} = X^*$. The corresponding topology $(X, \mathcal{U}_{X^*})$ is called **weak topology** *on X.*

Equivalently, x_α tends to x weakly (denoted $x_\alpha \rightharpoonup x$), if for all $x^* \in X^*$, $x^*(x_\alpha) \to x^*(x)$.

Proposition 4. *Weak topology is weaker than the strong topology, i.e. $\mathcal{U}_{X^*} \subset \mathcal{U}_{\|\cdot\|}$. That is, every weakly open set is strongly open. Equivalently,* **every strongly convergent sequence is weakly convergent***, i.e.*

$$\|x_\alpha - x\| \to 0 \Longrightarrow x_\alpha \rightharpoonup x.$$

Remarks: The generic converse is false.

- In $l^p, 1 < p < \infty$, $e_n \rightharpoonup 0$, while $\|e_n\| = 1$ (Exercise)
- In $C[0,1]$, $f_n \rightharpoonup f$, if and only if $\sup_n \|f_n\|_{C[0,1]} < \infty$ and f_n tends to f point-wise. (Exercise)
- In $l^p, 1 < p < \infty$, $x^n \rightharpoonup x$ if and only if $\sup_n \|x^n\|_{l^p} < \infty$ and $x^n_k \to x_k$ for each k. (Exercise)

Proposition 5. *If $\mathcal{U}_{X^*} = \mathcal{U}_{\|\cdot\|}$, then $dim(X) < \infty$.*

For example $S = \{x \in X : \|x\| = 1\}$ is always norm closed, if $dim(X) = \infty$, one has that S is not weakly closed, since 0 is in the weak closure of S. That is, if $dim(X) = \infty$, there is $x_\alpha : \|x_\alpha\| = 1$, so that $x_\alpha \rightharpoonup 0$.

The next result makes Proposition 5 ever more puzzling.

Theorem 15. *(Shur's theorem)*

In l^1, $\lim_n \|x^n - x\|_{l^1} = 0$ if and only if $x_n \rightharpoonup x$.

Note: Why is this not a contradiction with Proposition 5?

3.2 Weak$*$ topology on X^*

Definition 15. *(Weak$*$ topology on X^*)*

Let X real normed vector space, consider X^* and $\mathscr{F} = X$. The corresponding topology $(X^*, \mathscr{U}_X)$ is called **weak$*$ topology** *on X^*.*

Equivalently, x^*_α tends to x^* weak$*$ (denoted $x^*_\alpha \rightharpoonup x^*$), if for all $x \in X$, $\langle x^*_\alpha, x\rangle \to \langle x^*, x\rangle$.

Remark: If X is reflexive[2], then weak and weak$*$ topologies on X^* coincide.

Proposition 6. *On X^*, the weak$*$ topology, $(X^*, \mathscr{U}_X)$ is weaker than the weak topology $(X^*, \mathscr{U}_{X^{**}})$, which is weaker than the norm topology.*

Proposition 7. *Let X be a normed space and $K \subseteq X$ is a convex subset. Then K is closed if and only if K is weakly closed.*

Lemma 5. *(Mazur's lemma)*

Let X be a normed space and $x_n \rightharpoonup x$. Then, $x \in \overline{conv\{x_n\}}$. Equivalently, for all $\epsilon > 0$, there exists $N, \lambda_1 \geq 0, \ldots, \lambda_N \geq 0 : \sum_{j=1}^N \lambda_j = 1$, so that $\|x - \sum_{j=1}^N \lambda_j x_j\|_X < \epsilon$.

Lemma 6. *Let $x_n \rightharpoonup x$ or $x_n \rightharpoonup x$. Then, $\sup_n \|x_n\| < \infty$ and*

$$\|x\| \leq \liminf_n \|x_n\|.$$

For Hilbert spaces or more generally locally convex spaces, there is the partial reverse as follows.

Proposition 8. *Let X - Hilbert space or more generally locally convex space[3]. The following are equivalent.*

1. $x_n \rightharpoonup x$ *and* $\lim_n \|x_n\| = \|x\|$.
2. $\lim_n \|x_n - x\|_X = 0$.

[2] which happens exactly when X^* is reflexive

[3] It turns out that these are always reflexive so the weak and weak$*$ topologies coincide

3.3 Banach-Alaoglu's theorem

Here is a version of the Banach-Alaoglu's theorem, which is particularly useful in the applications.

Theorem 16. *Assume that X is a separable Banach space. Then, every bounded sequence in X^* has a weak$*$ convergent subsequence. In other words, every bounded set is weak$*$ pre-compact*[4].

The full theorem is as follows.

Theorem 17. *(Banach-Alaoglu's theorem)*

The unit ball B_{X^*} is weak$*$ compact. In other words, $(B_{X^*}, \mathcal{U}_X)$ is compact.

Remark: The theorem fails for the weak topology, unless the weak$*$ topology coincides with the weak topology. In fact $(B_{X^*}, \mathcal{U}_{X^{**}})$ is compact if and only if X is reflexive if and only if X^* is reflexive.

4 Fredholm theory

Definition 16. *Let X, Y be normed spaces and $A : X \to Y$ be a bounded linear operator. Define $A^* : Y^* \to X^*$ by the assignment*

$$\langle A^* y^*, x \rangle = \langle y^*, Ax \rangle$$

We use the

Lemma 7. *Let $A : X \to Y$ is a bounded linear operator. Then, $A^* \in B(Y^*, X^*)$ and $\|A^*\| = \|A\|$.*

Lemma 8. *Let $A : X \to Y$, $B : Y \to Z$. Then $(BA)^* = A^* B^*$ and $I^* = I$.*

[4] Technically speaking, the claim is about weak$*$ sequential pre-compactness. In view of the assumption that X is separable, these two notions are the same.

Examples:

1. $A \in M_{n\times m}$, $A : \mathbf{R}^m \to \mathbf{R}^n$, $A = (a_{ij})_{1\le i\le n, 1\le j\le m}$,
$$(Ax)_i = \sum_{j=1}^{m} a_{ij} x_j.$$
$A^t = (a_{iji})_{1\le i\le n, 1\le j\le m}$, $A^t : \mathbf{R}^n \to \mathbf{R}^m$.

2. $A : l^2 \to l^2$, $(Ax)_n = x_{n+1}, n = 1, \ldots$.
Then, $(A^* y)_1 = 0$, $(A^* y)_n = y_{n-1}, n = 2, \ldots$.

Theorem 18. *Let $A : X \to Y$. Then,*

1. $Im(A)^\perp = Ker(A^*)$
2. ${}^\perp Im(A^*) = Ker(A)$
3. *A has dense range if and only if A^* is injective.*

Note that if $Im(A)$ is closed, then $Im(A) = {}^\perp Im(A)^\perp = Ker(A^*)^\perp$.

Example: $A : l^2 \to l^2$, $(Ax)_n = \frac{x_n}{n}$ is bounded operator, with dense image, but $Im(A) \subsetneq l^2$.

Lemma 9. *Let $A : X \to Y$, $x^* \in X^*$. Then, the following are equivalent (TFAE)*

1. $x^* \in Im(A^*)$
2. *There exists $c > 0$, so that for each $x \in X$,*
$$|\langle x^*, x\rangle| \le c\|Ax\|_Y.$$

4.1 Algebraic factorization of maps through invertible maps

Let X, Y be vector spaces and $A : X \to Y$ be linear map. Not all maps are (algebraically) invertible, in fact A must be injective and surjective in order to be invertible.

Introduce $\pi : X \to X_0 := X/Ker(A)$, defined $\pi(x) = [x]$ (where $[x_1] = [x_2]$ if $x_1 - x_2 \in Ker(A)$). Then, one can define $A_0 : X_0 \to Y$.

$$A_0([x]) := Ax.$$

Clearly, this definition is independent on the representative. Also, $A_0 : X_0 \to Y_0 := Im(A)$. Such a map is clearly invertible (A_0 has $Ker(A_0) = \{0\}$, and $Im(A_0) = Im(A) = Y_0$). Furthermore, the inclusion $Y_0 = Im(A) \subseteq Y$ is denoted by i. So, one can factorize $A = i \circ A_0 \circ \pi$.

Question: When is such a map A_0 continuous? When is its inverse continuous? What does it mean in terms of estimates?

The following proposition provides the answers.

Proposition 9. *Let X, Y be vector spaces and $A : X \to Y$ be linear map. Then, one has the factorization*

$$A = i \circ A_0 \circ \pi.$$

where $A_0 : X/Ker(A) \to Im(A)$ is invertible.

Suppose now that X, Y are normed vector spaces. If A is bounded, then $A_0 : X_0 \to Y_0$ is bounded as well and

$$\|A_0\|_{B(X_0,Y_0)} = \sup_{x \in X} \frac{\|A(x)\|_Y}{\inf_{\xi \in Ker(A)} \|x + \xi\|_X} \le \|A\|_{B(X,Y)}.$$

Theorem 19. *(Closed Image Theorem)*

Let $A : X \to Y$ be bounded linear map, $A^* : Y^* \to X^*$. TFAE

1. $Im(A) = {}^{\perp} Ker(A^*)$.
2. $Im(A)$ *is closed.*
3. *there exists $c > 0$, so that for all $x \in X$,*

$$\inf_{\xi \in Ker(A)} \|x + \xi\|_X \le c\|Ax\|_Y. \tag{1}$$

4. $Im(A^*) = Ker(A)^{\perp}$

5. *$Im(A^*)$ is closed.*

6. *there exists $c > 0$, so that for all $x^* \in X^*$,*

$$\inf_{\xi^* \in Ker(A^*)} \|x^* + \xi^*\|_{X^*} \le c\|A^* x\|_{Y^*}.$$

Remark: These are all equivalent to $A_0 : X_0 \to Y_0$ has bounded inverse.

4.2 Some important corollaries from the Closed Image Theorem

Proposition 10. *(see Corollary 4.1.17/page 172)*

Let $A : X \to Y$, X, Y-Banach. Then

- *A is surjective if and only if A^* is injective and $Im(A^*)$ is closed. Equivalently,*

$$\|y^*\|_{Y^*} \le c\|A^* y^*\|_{X^*}. \tag{2}$$

- *A^* is surjective if and only if A is injective and $Im(A)$ is closed. Equivalently,*

$$\|x\|_X \le c\|Ax\|_Y. \tag{3}$$

A simple consequence is

Corollary 8. *Let $A : X \to Y$ be a bounded linear operator. Then, A is a bijection if and only if A^* is a bijection.*

4.3 Compact operators

Definition 17. *We say that $K : X \to Y$ is* **compact operator***, if $K(B_X)$ is precompact. Equivalently, for every bounded sequence x_n, $\{Kx_n\}$ has a convergent subsequence. In particular, K is bounded.*

A bounded operator $A : X \to Y$ is called a **finite rank operator***, if $dim(Im(A)) < \infty$. Note that each finite rank operator is compact.*

Lemma 10. *(Lemma 4.2.3/page 174)*

Let $K : X \to Y$ be compact. Then, if $x_n \rightharpoonup x$, then $\lim_n \|Kx_n - Kx\| = 0$.

Exercise: Show that A is finite rank if and only if there exists $x_1^*, \dots, x_N^* \in X^*$ and $y_1, \dots, y_N \in Y$, so that $A = \sum_{j=1}^N x_j^* \otimes y_j$ or

$$Ax = \sum_{j=1}^N \langle x_j^*, x \rangle y_j.$$

Also, check the exercises/examples on page 175.

Theorem 20. *(Theorem 4.2.10/page 175) We have the following.*

1. *Let $A : X \to Y$ and $B : Y \to Z$ be both bounded. If one of them is compact, then $BA : X \to Z$ is compact as well.*

2. *$K_n : X \to Y$ are compact and* $\lim_n \|K_n - K\| = 0$. *Then, K is compact as well.*

3. *K is compact if and only if K^* is compact.*

4.4 Fredholm operators

Definition 18. *Let $A : X \to Y$ be a bounded linear operator. As usual*

$$Ker(A) = \{x : Ax = 0\} \subset X; Im(A) = \{Ax : x \in X\} \subseteq Y, coKer(A) := Y / Im(A).$$

If $Im(A)$ is a closed subspace, then $coKer(A)$ is a Banach space.

A is Fredholm, if $dim(Ker(A)) < \infty, dim(CoKer(A)) < \infty$. In this case, we introduce its index

$$ind(A) = dim(Ker(A)) - dim(CoKer(A)).$$

Examples:

1. If $dim(X), dim(Y) < \infty$, then any $A : X \to Y$ is Fredholm and

   $$ind(A) = dim(X) - dim(Y)$$

   For $X = Y = \mathbf{R}^n$ and $A \in M_{n,n}$, this says $ind(A) = 0$. This is the rank-nullity theorem (i.e. $dim(Ker(A)) + dim(Im(A)) = n$) in disguise. Why?

2. if $A : X \to Y$ is a bijection, then $ind(A) = 0$.

Next is the duality theorem, which states that Fredholmness if preserved under taking adjoints.

Theorem 21. *Let $A : X \to Y$ is bounded operator. Then,*

- *If A, A^* have closed images, then*

$$dim(Ker(A^*)) = dim(CoKer(A)), dim(CoKer(A^*)) = dim(Ker(A)).$$

- *A is Fredholm if and only if A^* is Fredholm. In that case,*

$$ind(A) = -ind(A^*).$$

4.5 Some motivations

The first one is about matrices and a basic question in linear algebra. Let $A \in M_{m,n}$, so that $A : \mathbf{R}^n \to \mathbf{R}^m$, everything is real.
Question: When is the following linear equation

$$Ax = b, x \in \mathbf{R}^n, b \in \mathbf{R}^m \tag{4}$$

solvable? This is clearly solvable if and only if $b \in Im(A)$. In finite dimensions, all subspaces are closed, so we can use Theorem 20 to say $Im(A) = Ker(A^*)^\perp$. So, (4) is solvable if and only if $b \perp Ker(A^*)$. More specifically, we proved

Proposition 11. *The equation* (4) *is solvable if and only if $b \perp Ker(A^*)$. In particular, if $Ker(A^*) = \{0\}$, then* (4) *is solvable for each $b \in \mathbf{R}^m$.*

The other example is more sophisticated and it involves operator equations in the form

$$(Id - K)f = g, \tag{5}$$

where $K : X \to X$ is a **compact operator**, X is a Banach spaces. We will show later that such operators are Fredholm of index zero. In particular, $Im(I - K)$ is closed. So,

Proposition 12. *The equation* (5) *is solvable if and only if* $g \perp Ker(I-K^*)$*. If* $Ker(I-K^*) = \{0\}$*, then* (5) *is uniquely solvable.*

Proof. Again, $Im(I-K)$ is closed and so, $Im(I-K) = Ker(I-K^*)^{\perp}$. If $Ker(I-K^*) = \{0\}$, then $Im(I-K) = X$, whence $CoKer(I-K) = \{0\}$, so $codim(I-K) = 0$. Since $ind(I-K) = 0$, it follows that $dim(Ker(I-K)) = 0$, so $Ker(I-K) = \{0\}$. It follows that $I-K$ is a bijection and (5) is uniquely solvable. □

The results in Propositions Propositions 11 and 12, are typical to what is referred to as Fredholm alternatives.

Example: Let $K : [0,1] \times [0,1] \to \mathscr{C}$ is a continuous function. Then, the operator

$$\mathscr{K} f(x) = \int_0^1 K(x,y) f(y) dy : L^2[0,1] \to L^2[0,1],$$

is compact. Thus, an equation in the form[5]

$$\lambda f(x) - \int_0^1 K(x,y) f(y) dy = g(x), \tag{6}$$

where $\lambda \in \mathscr{C}$ and $g \in L^2[0,1]$ has an unique solution if and only if $Ker(\bar{\lambda} - \mathscr{K}^*) = \{0\}$ or

$$\bar{\lambda} z(x) = \int_0^1 \bar{K}(y,x) z(y) dy,$$

has no solutions $z \in L^2[0,1]$. Why? Note the reversed role of $(x,y) \to (y,x)$, this is not a typo.

4.6 Characterization of Fredholm operators

We start with an important lemma in the theory, which may be of independent interest.

Lemma 11. *(Lemma 4.3.9) Let* $D : X \to Y$*,* X, Y *are Banach spaces.*

Then, D *has a finite dimensional kernel and closed image* <u>*if and only if*</u> *there exists a Banach space* Z *and a compact operator* $K : X \to Z$*, so that*

$$\|x\|_X \leq c(\|Dx\|_Y + \|Kx\|_Z). \tag{7}$$

[5]This is a toy version of something that appears very frequently in PDE theory!

Here is the characterization of the Fredholmness. Roughly speaking, Fredholm operators are those that are invertible modulo compacts. *Note the typo in the book, the partial inverse F, must be* $F : Y \to X$ *instead of* $F : X \to Y$.

Theorem 22. *(Theorem 4.3.8)*

Let $A : X \to Y$ *be a bounded linear operator,* X, Y *are Banach spaces. Then, the following are equivalent.*

1. *A is Fredholm*
2. *There exists a bounded linear operator* $F : Y \to X$*, so that* $Id_X - FA : X \to X$*,* $Id_Y - AF : Y \to Y$ *are both compacts.*

Comments:

1. From the proof, it is clear that it is enough to assume that there are two (maybe different ones), $F_1, F_2 : Y \to X$, so that $I_X - F_1 A$ and $Id_Y - AF_2$ are compacts. This is sometimes useful.
2. By Theorem 20, $K(X) = \{K : X \to X, K - \text{compact}\}$ is a closed, two side ideal, in the algebra $B(X)$. Thus, one may define the Calkin algebra
$$L(X) = B(X)/K(X),$$
In it, one may state the theorem as follows A is Fredholm if and only if $[A]$ is invertible in the Calkin algebra (with inverse $[F]$ as in the theorem).
3. For every compact operator $K : X \to X$, operators of the type $Id - K : X \to X$ is Fredholm. Just apply the theorem with $F = Id$.

Exercise: If $B : X \to X$ is invertible and $K : X \to X$ is compact, then $B - K$ is Fredholm. The main focus this week is on how to compute the index.

4.7 Composition of Fredholm operators

Theorem 23. *(Theorem 4.4.1) Let $A : X \to Y$ and $B : Y \to Z$ are both Fredholm. Then, $BA : X \to Z$ is Fredholm and*

$$ind(BA) = index(A) + ind(B)$$

4.8 Stability of the Freholm index

The following result is a basic result in the theory, stating that a small (in norm) perturbation does not change the index and neither does perturbation by compact (even it is a large compact operator).

Theorem 24. *(Theorem 4.4.2) Let $D : X \to Y$ is Fredholm. Then*

1. *If $K : X \to Y$ is compact, then $D + K$ is Fredholm and*

$$ind(D+K) = ind(D)$$

   In other words, adding compact to a Fredholm preserves the Fredholmness and keeps the index unchanged.

2. *There exists a constant $\epsilon > 0$, so that whenever $P : X \to Y$ is a bounded operator, $\|P\| < \epsilon$, then $D + P$ is Fredholm and $ind(D+P) = ind(D)$.*

   In other words, the Fredholm operators are an open set in the space of bounded operators $B(X,Y)$ and $ind : Fredholm \to \mathbb{N}$ is locally a constant. In fact,

$$Fredholms = \cup_{n\in\mathbb{Z}} \Omega_n,$$

   where each set $\Omega_n = \{A : X \to Y, ind(A) = n\}$ is a component in the set of Fredholm operators.

4.9 Applications

Immediately from the previous results, for each $K : X \to X$ compact, we have $ind(I+K) = ind(I) = 0$.

Theorem 25. *Let $K : X \to X$ be a compact operator. Then, for each $\lambda \in \mathbb{C}, \lambda \neq 0$,*

1. $dim(Ker(\lambda - K)) < \infty$.
2. $dim(Ker(\lambda - K)) = dim(Ker(\lambda - K^*))$
3. *Suppose* $Ker(\lambda - K) = \{0\}$. *Then,* $(\lambda Id - K)$ *is invertible.*
4. *(Fredholm alternative) The equation*

$$(\lambda - K)f = g, f, g \in X \tag{8}$$

has solutions if and only if $g \perp Ker(\lambda - K^*)$. *More specifically, If* $Ker(\lambda - K) = \{0\}$, *then* (8) *is uniquely solvable, in fact* $f = (\lambda - K)^{-1}g$.

If $Ker(\lambda - K) \neq \{0\}$, *let* $n = dim(Ker(\lambda - K)) \geq 1$. *There exist* $x_1, \ldots, x_n \in X$, *a basis of* $Ker(\lambda - K)$ *and* $x_1^*, \ldots, x_n^* \in X^*$ *a basis of* $Ker(\lambda - K^*)$, *so that* (8) *has solutions if and only if* $\langle x_j^*, g \rangle = 0$. *If this is satisfied,* $g \in Im(\lambda - K)$, $(\lambda - K) : X \to Im(\lambda - K)$ *is invertible and the general solution of* (8) *is in the form*

$$f = (\lambda - K)^{-1}g + \sum_{j=1}^{n} \mu_j x_j.$$

Remark: The condition $\lambda \neq 0$ is crucial. The theorem fails for $\lambda = 0$.

5 Spectral theory

The first part is section 5.1 in the book (pages 198-202), which introduces the concept of a Banach space over the complex numbers. I have mentioned several times how various things work in that case. Bottom line is that all the theorems remain the same, including property of norms, linear functionals, Hahn-Banach[6], open mapping theorem, closed graph theorem, inverse function theorem, Banach-Alaoglu theorem, Fredholm theory. I recommend you read these pages on your own.

[6] which needs a little extension from the real case, which I mentioned about in class

5.1 Integration

We investigate the following:

Question: How does one integrate Banach space valued (B - valued) functions? How much regularity does one need and what it means for B - valued functions?

A class that will be enough for our purposes is continuous functions, although there is a notion of Riemann/Lebesgue integrable B - valued functions.

Lemma 12. *(Integration of continuous functions, page 202) Let X be a Banach space (real or complex) and $x : [a, b] \to X$ be a continuous B - valued function. Then, there exists an unique $\xi \in X$, so that "$\xi = \int_a^b x(t)dt$". Formally, for every $x^* \in X^*$,*

$$\langle x^*, \xi \rangle = \int_a^b \langle x^*, x(t) \rangle dt.$$

The approach above is very common in how one passes from B - space valued functions to "regular" scalar valued functions. The next lemma is a prime example.

Lemma 13. *Let X be a Banach space (real or complex), $x, y : [a, b] \to X$ are continuous B - valued functions. Then,*

1. *(Linearity of the integral)*
$$\int_a^b (x(t) + cy(t))dt = \int_a^b x(t)dt + c\int_a^b y(t)dt.$$
2. *For $a < c < b$,*
$$\int_a^b x(t)dt = \int_a^c x(t)dt + \int_c^b x(t)dt.$$
3. *Let $A : X \to Y$ be bounded linear operator, then*
$$A\int_a^b x(t)dt = \int_a^b Ax(t)dt.$$
4. *(Fundamental theorem of calculus)*

   Let $x : [a, b] \to X$ be continuously differentiable, i.e. there is a continuous B - valued function $g(t)$, so that $\lim_{h\to 0} \|\frac{x(t+h)-x(t)}{h} - g(t)\| = 0$ (and then $\dot{x}(t) = g(t)$). Then,
$$\int_a^b \dot{x}(t)dt = x(b) - x(a).$$

5. *(change of variables formula)*

   Let $\phi : [\alpha, \beta] \to [a, b]$ be a differentiable and invertible transformation. Then

$$\int_a^b \dot{x}(t)dt = \int_\alpha^\beta x(\phi(s))\phi'(s)ds.$$

6. *(Triangle inequality for integrals)*

$$\|\int_a^b x(t)dt\| \leq \int_a^b \|x(t)\|dt$$

7. *If*

$$x(t) = x_0 + \int_a^t y(s)ds,$$

   then x is continuously differentiable with $\dot{x}(t) = y(t)$.

Next topic is holomorphic B valued functions.

5.2 Holomorphic functions

Definition 19. *Let $\Omega \subset \mathbb{C}$ be an open set, X is a complex Banach space, $f : \Omega \to X$ is continuous. We say that f is holomorphic, denoted $H(\Omega)$, if*

$$f'(z) := \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \in X,$$

exists and $f' : \Omega \to X$ is continuous function.

Let γ be a C^1 curve in Ω, i.e. $\gamma : [a, b] \to \Omega$, $\gamma \in C^1(a, b) \cap C[a, b]$. Then,

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(s))\gamma'(s)ds.$$

It is tempting to ask:

Question: Is it true that f is B valued holomorphic function if and only if $z \to \langle x^*, f(z)\rangle$ is (scalar) holomorphic for each $x^* \in X^*$

The next lemma shows that and it is in fact more general.

Lemma 14. *Let $\Omega \subset \mathbb{C}$ be an open set, X, Y are complex Banach spaces and $A : \Omega \to L(X, Y)$ is a weakly continuous function, i.e. for each $x \in X, y^* \in Y^*$, $z \to \langle y^*, A(z)x\rangle$ is continuous function.*

Then, the following are equivalent:

1. *A is holomorphic.*

2. *For each $x \in X, y^* \in Y^*$,*
$$z \to \langle y^*, A(z)x\rangle$$
is holomorphic on Ω.

3. *(Cauchy theorem) For each $z_0 \in \Omega$ and $r_0 > 0$, so that $\overline{B_r(z_0)} \subset \Omega$,*
$$\langle y^*, A(\omega)x\rangle = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{\langle y^*, A(z)x\rangle}{z-\omega} dz, \tag{9}$$
for each $r : 0 < r < r_0$ and $\omega : |\omega - z_0| < r$.

Remarks:

- Formula (9) may be generalized to closed curves of index one around z_0.
- If you apply this to the mapping $A(z) = f(z)x^*$, where $f : \Omega \to Y$ and x^* is arbitrary non-zero element of X^*, we obtain that $f : \Omega \to Y$ is holomorphic if and only if f is weakly holomorphic, i.e. for every $y^* \in Y^*$, $z \to \langle y^*, f(z)\rangle$ is holomorphic. Also, applying (9) to this, we obtain the Cauchy formula
$$f(\omega) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{z-\omega} dz. \tag{10}$$

Some exercises that are good results.

Proposition 13. *Let X be a Banach space, $f : \Omega \to X$ be holomorphic. Then,*

1. *$f' : \Omega \to X$ is also holomorphic.*

2.

$$f^{(n)}(\omega) = \frac{n!}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-\omega)^{n+1}} dz. \tag{11}$$

3. Let $\{a_n\}_n$ *is a sequence of elements in* X*. Suppose that* $\limsup_{n\to\infty} \|a_n\|^{\frac{1}{n}} < \infty$*, so that*

$$\rho := \frac{1}{\limsup_{n\to\infty} \|a_n\|^{\frac{1}{n}}} > 0.$$

Then, the formula

$$f(z) := \sum_{n=0}^{\infty} a_n z^n,$$

defines a B *valued function on the domain* $B_\rho(0) \subset \mathbb{C}$*. Also,*

$$a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}} dz, r < \rho.$$

The function that we really want to consider is, for $A \in L(X)$, the operator valued function $z \to (z - A)^{-1}$, whenever it exists. This is called resolvent set $\rho(A)$, which happens to be open subset of $\mathbb{C}$. It is also the complement of the spectrum $\sigma(A)$.

5.3 Spectrum

Definition 20. *Let* X *be a complex Banach space and* $A \in L(X)$ *be bounded linear operator.* **The spectrum** *is introduced as follows*

$$\sigma(A) = \{\lambda \in \mathbb{C} : (\lambda - A) \text{ is not bijective/invertible}\}$$

Equivalently **the resolvent set** $\rho(A) = \mathbb{C} \setminus \sigma(A)$ *can be defined independently as*

$$\rho(A) = \{\lambda \in \mathbb{C} : (\lambda - A) \text{ is bijective/invertible}\}$$

Question: What are the reasons for $\lambda - A$ not to be invertible? It is either not injective or not surjective or both. We then subgroup them in two distinct/disjoint subgroups as follows - it is either A) not injective, or B) it is injective, but not surjective.

You can further group them in three distinct major groups(but be aware that there are other ways to group them, which makes it so confusing!) - basically group B is split in additional two groups. More specifically:

1. Non-trivial kernel - $Ker(\lambda - A) \neq \{0\}$, i.e. $\lambda - A$ is not injective.
2. $Ker(\lambda - A) = \{0\}$, but $Im(\lambda - A)$ is not dense in X.
3. $Ker(\lambda - A) = \{0\}$, $Im(\lambda - A)$ is dense in X, but $Im(\lambda - A) \neq X = \overline{Im(\lambda - A)}$.

The reasons for this split are historical and are partially motivated by the stability of these parts of the spectrum under perturbations.
Eventually $\sigma(A) = P\sigma(A) \cup R\sigma(A) \cup C\sigma(A)$, as follows.

Definition 21.

- **Point spectrum** - *$\lambda - A$ is not injective or $Ker(\lambda - A) \neq \{0\}$.*

$$P\sigma(A) = \{\lambda \in \mathbb{C} : Ker(\lambda - A) \neq \{0\}\}.$$

- **Residual spectrum** *$Ker(\lambda - A) = \{0\}$, but $Im(\lambda - A)$ is not dense in X.*

$$R\sigma(A) = \{\lambda \in \mathbb{C} : Ker(\lambda - A) = \{0\}, \overline{Im(\lambda - A)} \neq X\}.$$

- **Continuous spectrum** *$Ker(\lambda - A) = \{0\}$, $\overline{Im(\lambda - A)} = X$, but the image is not closed, or $Im(\lambda - A) \neq \overline{Im(\lambda - A)} = X$.*

$$C\sigma(A) = \{\lambda \in \mathbb{C} : Ker(\lambda - A) = \{0\}, Im(\lambda - A) \neq \overline{Im(\lambda - A)} = X\}.$$

Remarks:

- In finite dimensions $A : X \to X$, $dim(X) = n$, $\sigma(A) = P\sigma(A)$, while $R\sigma(A) = C\sigma(A) = \emptyset$. Moreover, $\sigma(A)$ is a finite set

$$\#\sigma(A) \leq n$$

This is just the fact that a matrix $\lambda - A$ is non-invertible if and only if $\det(\lambda - A) = 0$. λ is then called an eigenvalue. In this case,

$$\sigma(A) = \{\lambda : \det(\lambda - A) = 0\}, \#\sigma(A) = n,$$

if one counts eigenvalues with respective multiplicity.

- $X = l^2$. For the shift operators $A, B : l^2 \to l^2$ defined by $Ax = (x_2, x_3, \ldots), Bx = (0, x_1, \ldots)$, we have

$$\begin{aligned}
\sigma(A) &= \{\lambda : |\lambda| \le 1\}, P\sigma(A) = \{\lambda : |\lambda| < 1\}, R\sigma(A) = \emptyset, C\sigma(A) = \{\lambda : |\lambda| = 1\} \\
\sigma(B) &= \{\lambda : |\lambda| \le 1\}, P\sigma(B) = \emptyset, R\sigma(B) = \{\lambda : |\lambda| < 1\}, C\sigma(B) = \{\lambda : |\lambda| = 1\}
\end{aligned}$$

Remark: For A, eigenvectors are $x_\lambda = (\lambda, \lambda^2, \ldots), |\lambda| < 1$, so that $Ax_\lambda = \lambda x_\lambda$. For B, clearly $(\lambda - B)x = 0$ implies $x = 0$.

Lemma 15. *Let $A : X \to X$ be a bounded linear operator, $A^* : X^* \to X^*$ is the dual operator. Then*

- $\sigma(A)$ *is a compact subspace of* $\mathbb{C}$
- $\sigma(A^*) = \sigma(A)$
- $$\begin{aligned}
&P\sigma(A^*) \subset P\sigma(A) \cup R\sigma(A), R\sigma(A^*) \subset P\sigma(A) \cup C\sigma(A), C\sigma(A^*) \subset C\sigma(A) \\
&P\sigma(A) \subset P\sigma(A^*) \cup R\sigma(A^*), R\sigma(A) \subset P\sigma(A^*), C\sigma(A) \subset R\sigma(A^*) \cup C\sigma(A^*).
\end{aligned}$$

Proof. For the boundedness of $\sigma(A)$, we show that $\sigma(A) \subset \{\lambda : |\lambda| \le \|A\|\}$. Indeed, let $\lambda : |\lambda| > \|A\|$. Then,

$$\lambda - A = \lambda(I - \lambda^{-1}A).$$

Since $\|\lambda^{-1}A\| = \frac{\|A\|}{|\lambda|} < 1$,, the operator $(I - \lambda^{-1}A)$ is invertible by von Neumann, see Lemma 20. It remains to show that $\sigma(A)$ is closed. It is equivalent to show that $\rho(A)$ is open. This is again Lemma 20. Indeed, let $\lambda_0 \in \rho(A)$. Then, for all $\lambda : |\lambda - \lambda_0| \le \frac{1}{\|(\lambda_0 - A)^{-1}\|}$, we have by Lemma 20 that

$$\lambda - A = (\lambda - \lambda_0) + (\lambda_0 - A) = (\lambda_0 - A)(I + (\lambda - \lambda_0)(\lambda_0 - A)^{-1}).$$

Again, by von Neumann, if $\|(\lambda - \lambda_0)(\lambda_0 - A)^{-1}\| < 1$, we have invertibility of $\lambda - A$, so $\rho(A)$ is an open set.

2) follows from Corollary 8, which states that $\lambda \in \rho(A)$ if and only if $\lambda - A$ is bijection, if and only if $\lambda - A^*$ is a bijection if and only if $\lambda \in \rho(A^*)$. □

Next, we discuss the resolvent of A, whenever it is defined.

Lemma 16. *Let X be a complex Banach space and $A \in B(X)$. Then $\rho(A)$ is an open set and define, for $\lambda \in \rho(A)$,*

$$R_\lambda(A) := (\lambda - A)^{-1}$$

Then $R : \rho(A) \to B(X)$ is a holomorphic $B(X)$ valued mapping, which satisfies **the resolvent identity**

$$R_\lambda(A) - R_\mu(A) = (\mu - \lambda)R_\lambda(A)R_\mu(A), \lambda, \mu \in \rho(A). \tag{12}$$

In particular, its complex derivative satisfies

$$R'(\lambda) = -R_\lambda(A)^2.$$

Remark: Note that the resolvents commute, i.e. $R_\lambda(A)R_\mu(A) = R_\mu(A)R_\lambda(A)$, if we apply (12) with μ instead of λ and λ instead of μ.

We first establish basic properties of spectra.

5.4 Spetral radius formula

We first define spectral radius.

Definition 22. *For a bounded operator $A \in L(X)$, define its spectral radius*

$$r_A := \inf\{\mu > 0 : \sigma(A) \subset \{z : |z| < \mu\}\} = \sup_{\lambda \in \sigma(A)} |\lambda| \geq 0.$$

Theorem 26. *Let X be non-trivial complex Banach space and $A \in L(X)$. Then, $\sigma(A)$ is non-empty and*

$$r_A = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

Remark: Part of the claim is that the limit $\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$ exists.

5.5 Spectrum of compact operators

Most of the stuff here is already contained in Theorem 25, but some of the notions come up for more general operators. Clearly,

$$Ker((\lambda I - A)) \subseteq Ker((\lambda I - A)^2) \subseteq \ldots \subseteq Ker((\lambda I - A)^k) \subseteq Ker((\lambda I - A)^{k+1}) \subseteq \ldots$$

Here, the elements $Ker((\lambda I - A))$ are called eigenvectors, while the elements of $Ker((\lambda I - A)^2)$ are all elements, which are either eigenvectors or (first generations) adjoint eigenvectors . Think Jordan normal forms

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

so that $e_1 \in Ker(\lambda - J)$, while $e_2 \in Ker((\lambda - J)^2) \setminus Ker(\lambda - J)$ and so on. *The generalized eigenspace* is

$$E_\lambda(A) = \cup_{k=1}^{\infty} Ker((\lambda I - A)^k).$$

If $Ker((\lambda I - A)^k) = Ker((\lambda I - A)^{k+1})$, we say that it stabilizes and in fact $E_\lambda = Ker((\lambda I - A)^k)$. This does not have to happen for general A!

Theorem 27. *(Spectrum of compact operators) Let X be a complex Banach space and A is a compact operator on it. Then, $\sigma(A) \setminus \{0\} \subset P\sigma(A)$ and*

1. *For any $\lambda \in \sigma(K) \setminus \{0\}$, $dim(E_\lambda) < \infty$ and in fact there exists m, so that*

$$Ker((\lambda I - A)^m) = Ker((\lambda I - A)^{m+1})$$

   In such a case, $E_\lambda(A) = Ker((\lambda I - A)^m)$ and

$$X = E_\lambda \oplus Im((\lambda I - A)^m). \tag{13}$$

2. *Every non-zero eigenvalue is an isolated point in $\sigma(A)$.*

5.6 Holomorphic functional calculus

We would like to define $f(A) \in L(X)$ for every holomorphic function f defined on Ω, which contains $\sigma(A)$.

Definition 23. *For every curve $\gamma \subset \Omega$, so that $\sigma(A) \subset Int(\gamma)$, we require that for every $\lambda \in \sigma(A)$,*

$$ind_\gamma(\lambda) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\lambda} = 1.$$

Then, for every $f \in H(\Omega)$, introduce

$$f(A) := \frac{1}{2\pi i}\int_\gamma f(z)(z-A)^{-1}dz$$

Remark: By the Cauchy theorem, the choice of curve γ is non-essential, as long as the condition $ind_\gamma(\lambda) = 1$ for all $\lambda \in \sigma(A)$ is satisfied.

Theorem 28. *Let X be a complex Banach space and $A \in L(X)$. Then,*

1. *Let $\Omega \subset \mathbb{C}$, so that $\sigma(A) \subset \Omega$. Then, for every $f, g \in H(\Omega)$,*

$$(f+g)(A) = f(A) + g(A), (fg)(A) = f(A)g(A).$$

2. *$p(z) = \sum_{k=0}^N a_k z^k$, then $p(A) = \sum_{k=0}^n a_k A^k$.*

3. **Spectral mapping theorem**

$$\sigma(f(A)) = f(\sigma(A)).$$

4. *$f : \Omega \to U$, $g : U \to \mathbb{C}$, f, g holomorphic, then*

$$g(f(A)) = (g \circ f)(A).$$

5. *Let $\sigma(A) = \Sigma_0 \cup \Sigma_1$, where Σ_0 and Σ_1 are disjoint. Define the holomorphic functions $f_0|_{\Sigma_0} = 1$, $f_0|_{\Sigma_1} = 0$ and $f_1 = 1 - f_0$. Then, $P_0 := f_0(A)$, $P_1 = f_1(A)$ are projections, i.e. $P_j^2 = P_j, j = 1, 2$, $P_0P_1 = P_1P_0 = 0$ and they induce an A invariant decomposition $X = X_0 \oplus X_1$, $X_j = P_j(X)$, so that $A_j := (zf_j)(A) = AP_j : X_j \to X_j$ and $\sigma(A_j) = \Sigma_j$.*

5.7 Adjoint operators

We now specialize on operators on a Hilbert space. It is largely a repetition of the previously studied material, but now in the case of Hilbert spaces with complex scalars. *One* *subject, which is often subject of a lot of confusion, that I would like to discuss in detail, is adjoint versus dual operators.*

For complex matrices, on $\mathbf{R}^n$, we have two operations of adjoints - A^t and A^*. Namely, if $A = (a_{ij})_{i,j=1}^n$,

$$A^t = (a_{ji})_{i,j=1}^n, \ \ A^* = (\overline{a_{ji}})_{i,j=1}^n.$$

Thus, for the duality operation, $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$, we have $\langle Ax, y\rangle = \langle x, A^t y\rangle$, whereas for the dot product $(x, y) = \sum_{j=1}^n x_j \bar{y}_j$, we have $(Ax, y) = (x, A^* y)$. Similarly for Hilbert spaces, we have dual and adjoint operators. By Riesz representation theorem, one can define the duality operation in terms of the dot product, as follows

$$\langle x, y\rangle := (x, \bar{y})$$

Definition 24. *Let $H, (\cdot,\cdot)$ be a complex Hilbert space. Recall its dot product is sesquilinear, i.e. $(a, \lambda b) = \bar{\lambda}(a, b)$. Then, given $A \in L(H)$, the adjoint operator is defined as the unique operator A^*, with*

$$(Ax, y) = (x, A^* y).$$

From now on, whenever we talk about Hilbert spaces, A^* would mean the adjoint operator, rather than the dual one (I wish the standard notation were A^t for the dual, say in the previous chapter, but that is not the case!).

Next, here are some properties of the adjoint operator A^*.

Lemma 17. *(see Lemma 5.9, page 226) Let H be a Hilbert space and $A \in L(H)$. Then,*

- $A^* \in L(H)$ *and* $\|A^*\| = \|A\|$.
- $(AB)^* = B^* A^*$, $(\lambda A)^* = \bar{\lambda} A^*$.
- $A^{**} = A$.
- $Ker(A^*) = Im(A)^{\perp}$, $\overline{Im(A^*)} = Ker(A)^{\perp}$.
- $$\sigma(A^*) = \overline{\sigma(A)}.$$

5.8 Normal operators

Definition 25. *Let H be a complex Hilbert space and $A \in L(H)$. We say that*

- *A **is normal**, if $AA^* = A^*A$*
- *A **is unitary**, if $AA^* = A^*A = Id$ or $A^* = A^{-1}$.*
- *A **is self-adjoint**, if $A^* = A$.*

Note that every self-adjoint and unitary is normal as well.

Examples:

- Matrices $A = (a_{ij})_{i,j=1}^n$ with $a_{ij} = \bar{a}_{ji}$ are self-adjoint operators on $\mathbb{C}^n$.
- $A : L^2[0,1] \to L^2[0,1]$, $Ax(t) = f(t)x(t)$, where f is periodic continuous function. $M^*x(t) = \bar{f}(t)x(t)$. So, M is always normal and it is self-adjoint if and only if f is real-valued.
- The double infinite space $l^2 = \{x = \{x_n\}_{n=-\infty}^{\infty} : \|x\| = \left(\sum_{n=-\infty}^{\infty} |x_n|^2\right)^{\frac{1}{2}}\}$ and the shift operators $(Sx)(n) = x_{n+1}$, $(Rx)_n = x_{n-1}$. Note that $S^* = R, R^* = S$ and

$$SS^* = SR = Id = RS = S^*S.$$

  Thus, S, R are unitary operators on l^2.

Exercise 1. *Let A be self-adjoint. Then $A = 0$ if and only if $(Ax, x) = 0$ for all x.*

Proposition 14. *Let H be a complex Hilbert space and $A \in L(H)$. Then,*

- *A is normal if and only if $\|A^*x\| = \|Ax\|$ for all $x \in H$.*
- *A is unitary, if $\|Ax\| = \|A^*x\| = \|x\|$.*
- *A is self-adjoint, if $(Ax, x) \in \mathbf{R}$ for all $x \in H$.*

Theorem 29. *(Spectrum of normal operators) Let H be a complex Hilbert space and $A \in L(H)$ be a normal operator. Then,*

1. $\|A^n\| = \|A\|^n$ *for every* $n \in \mathbb{N}$.
2. $\|A\| = \sup_{\lambda \in \sigma(A)} |\lambda|$.
3. $R\sigma(A^*) = R\sigma(A) = \emptyset$, $P\sigma(A^*) = \{\bar{\lambda} : \lambda \in P\sigma(A)\}$.
4. *If A is unitary, then* $\sigma(A) \subset \{z : |z| = 1\}$.

Proof. 1) is easy, see page 229. 2) follows from it by the spectral radius formula. Part 3) is also well done there. □

Lemma 18. *Let $A = A^*$ on a Hilbert space and $\lambda \neq \mu \in P\sigma(A)$, with corresponding eigenvectors x, y, i.e. $Ax = \lambda x$, $Ay = \mu y$. Then, $(x, y) = 0$.*

Theorem 30. *(Characterization of normal compact operators) Let H be a complex Hilbert space and $A \in L(H)$ be a compact and normal operator. Then, there exists an orthonormal sequence $\{e_n\}_{n \in I}$ and $\lambda_i, i \in I$, so that $\lim_i \lambda_i = 0$ and*

$$Ax = \sum_{i \in I} \lambda_i (x, e_i) e_i.$$

5.9 Spectrum of a self-adjoint operators

Theorem 31. *Let H be a complex Hilbert space and $A \in L(H)$ be a self-adjoint operator. Then,*

1. $\sigma(A) \subset \mathbf{R}$
2. $\sup \sigma(A) = \sup_{\|x\|=1} (Ax, x)$
3. $\inf \sigma(A) = \inf_{\|x\|=1} (Ax, x)$.
4. $\|A\| = \sup_{x : \|x\|=1} |(Ax, x)|$.

We would like to extend the functional calculus, in the case of a self-adjoint operator, to the algebra of continuous functions. The idea is to construct, for a fixed self-adjoint operator A on a Hilbert space H, an algebra homomorphism $\Phi_A : C(\sigma(A)) \to L(H)$, so that $\Phi_A(e_0) = A$, where $e_0(x) = x$.

5.10 Banach algebras

Definition 26. *We say that a Banach space A is an unital Banach algebra, if in addition to the operations addition and multiplication by scalar, there are the operations product $(a,b) \to ab$ (and the unit element $\mathbb{1} : a\mathbb{1} = \mathbb{1}a = a$) and the star operation $a \to a^*$, with the properties*

$$\|ab\| \le \|a\|\|b\|,\ (ab)^* = b^* a^*,\ \mathbb{1}^* = \mathbb{1},\ (\lambda a)^* = \bar{\lambda} a^*,\ a^{**} = a,$$

and the star property

$$\|a^* a\| = \|a\|^2.$$

A is called commutative, if $ab = ba$ for all $a, b \in A$.

Examples:

- $C(K)$ - the continuous functions on a compact space K is commutative C^* algebra.
- $L(H)$, with the regular $*$ operation of adjoints. This is highly non-commutative Banach algebra.
- the space $l^1(\mathbb{Z}) = \{(x_n)_{n=-\infty}{}^{\infty} : \|x\| = \sum_{n=-\infty}^{\infty} |x_n|\}$, with the regular operations and a product operation given by $z = x.y$

  $$z_k = \sum_{n=-\infty}^{\infty} x_{k-n} y_n.$$

  is a commutative Banach algebra (Check!).

5.11 Continuous functional calculus for self-adjoint operators

The next lemma begins to build the homomorphism Φ_A, namely with the polynomials.

Definition 27. *Let A, B be C^* algebras, and $\Phi : A \to B$. We say that Φ is C^* algebra homomorphism, if*

$$\Phi(\mathbb{1}_A) = \mathbb{1}_B, \Phi(ab) = \Phi(a)\Phi(b), \Phi(a^*) = \Phi(a)^*.$$

We now start building such Φ on the algebra of continuous functions on $\sigma(A)$, $C(\sigma(A))$.

5.11.1 The map Φ_A on polynomials

Lemma 19. *Let H be a complex Hilbert space, $A \in L(H)$. For every $p(z) = \sum_{k=0}^n a_k z^k$,*

$$p(A) = \sum_{k=0}^{n} a_k A^k \in L(H)$$

has the properties

$$(p+q)(A) = p(A) + q(A), pq(A) = p(A)q(A), p(\sigma(A)) = \sigma(p(A)), \|p(A)\| = \sup_{\lambda \in \sigma(A)} |p(\lambda)|.$$

The next technical result is the (general) Stone-Weierstrass theorem.

Theorem 32. *(Stone-Weierstrass) Let M be a Hausdorf compact space and $A \subset C(M)$ be a subalgebra, with the following properties*

1. $1 \in A$
2. *A separates points on M, i.e. for every $x, y \in M, x \neq y$, there exists $f \in A$, so that $f(x) \neq f(y)$.*
3. *A is closed under conjugation, i.e. if $f \in A$, then $\bar{f} \in A$.*

Then, A is dense in $C(M)$.

5.11.2 The construction of the map $\Phi_A : C(\Sigma) \to L(H)$

Theorem 33. *Let H be a complex Hilbert space, $A \in L(H)$ be self-adjoint operator, i.e. $A = A^*$. Let $\Sigma = \sigma(A)$. Then, there exists a bounded, complex linear operator*

$$\Phi_A : C(\Sigma) \to L(H), f \to f(A),$$

so that

- **Product** $\Phi_A(\mathbb{1}) = Id$, $fg(A) = f(A)g(A)$.
- **Conjugation** $\bar{f}(A) = f(A)^*$
- **Normalization** *If* $f(\lambda) = \lambda$, *then* $f(A) = A$
- **Isometry** $\|f(A)\| = \sup_{\lambda\in\Sigma} |f(\lambda)|$
- **Commutative** *If* $AB = BA$, *then* $f(A)B = Bf(A)$ *for all* $f \in C(\Sigma)$.
- **Image**

$$\mathscr{A} := \{f(A) : f \in C(\sigma) = \cap\{\mathscr{B} : \mathscr{B}\ C^*\ algebra \subset L(H), A \in \mathscr{B}\}$$

- **Eigenvector** *If* $Ax = \lambda x$, *for some* $\lambda \in \mathbb{C}$ *and* $x \in H$, *then* $f(A)x = f(\lambda)x$.
- **Spectrum** $f(A)$ *is normal for all* $f \in C(\Sigma)$, $\sigma(f(A)) = f(\sigma(A))$.
- **Composition** *For* $f \in C(\Sigma, \mathbf{R})$, $g \in C(f(\Sigma))$, $g \circ f(A) = g(f(A))$.

5.11.3 Square roots

Definition 28. *We say that* $A \in L(H) : A = A^*$ *is positive semi-definite, if* $\langle Ax, x\rangle \geq 0$. *We denote it* $A \geq 0$.

We have the following theorem.

Theorem 34. *Let* H *be a complex Hilbert space and* $A = A^*$. *Let* $f \in C(\sigma(A))$. *Then,*

1. $f(A) = f(A)^*$ *if and only if* $f(\sigma(A)) \subset \mathbf{R}$.
2. *Assume* $f(\sigma(A)) \subset \mathbf{R}$. *Then,* $f(A) \geq 0$ *if and only if* $f \geq 0$.
3. $A \geq 0$ *if and only if there exists* $B = B^*$, *so that* $A = B^2$.

Proof. The proof of 1) is easy, see p. 245. For 2), we have

$$\inf_{\|x\|=1} \langle f(A)x, x\rangle = \inf \sigma(f(A))) = \inf f(\sigma((A))) = \inf_{\lambda\in\sigma(A)} f(\lambda)$$

So, $f(\lambda) \geq 0$ if and only if $f(A) \geq 0$.

For 3), assume that $A \geq 0$. Then, $\sigma(A) \subset [0, \infty)$. Then, consider the function $f(\lambda) = \sqrt{\lambda}$, which is well-defined. Then, $B := f(A) \in L(H)$. Also, since $f^2(\lambda) = \lambda$,

$$B^2 = f^2(A) = A.$$

The reverse direction is easy, since if $A = B^2$,

$$\langle Ax, x \rangle = \langle B^2 x, x \rangle = \langle Bx, Bx \rangle = \|Bx\|^2 \geq 0.$$

□

A Finite dimensional subspaces of a Banach space

We show that every finite dimensional subspace of a (real) Banach space is isomorphic to $\mathbf{R}^n$.

Proposition 15. *Let X be a real Banach space and $x_1, \dots, x_n$ be a finite family of linearly independent vectors in X. Then, $X_n = \overline{span[x_1, \dots, x_n]}$ is a Banach space isomorphic to $\mathbf{R}^n$. In particular,*

$$c\|\sum_{j=1}^n \lambda_j x_j\|_X \leq \max_{1\leq j\leq n} |\lambda_j| \leq C\|\sum_{j=1}^n \lambda_j x_j\|_X \tag{14}$$

B Dual families

Th next Proposition provides the existence of a finite dual family of vectors.

Proposition 16. *(see Corollary 2.3.4) Let X be a real Banach space and $x_1, \dots, x_n$ be a finite family of linearly independent vectors in X. Then, there exists a family of vectors $x_1^*, \dots, x_n^* \in X^*$, so that*

$$\langle x_j^*, x_i\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Remark: This is a generalization of a corollary of Hahn-Banach, which we use frequently. Namely, for every $x \in X$, there is $x^* \in X^* : \|x^*\| = 1$, so that $\langle x^*, x\rangle = \|x\|$, see Corollary 6. Here, we have a family of vectors, whose norms are unknown, but keep in mind that it is usually the case that $\max_{1\leq j\leq n} \|x_j^*\|_{X^*}$ is growing with n.

C Finite dimensional subspaces/finite co-dimension subspace of a Banach space are complemented

The next result is almost an immediate consequence of Proposition 16, see also the construction of K in the proof of Lemma 11.

Proposition 17. *Let X be a Banach space and X_0 be a finite dimensional subspace of X. Then, X_0 is complemented. That is, there exists a closed linear subspace $Y \subset X$, so that*

$$X = X_0 \oplus Y.$$

Equivalently, there is a projection operator $P : X \to X_0$, i.e. an $P^2 = P$. Note that $Q = I - P$ also satisfies $Q^2 = Q$ and then, $Y = Im(Q)$.

Let Y be a finite co-dimension subspace of X, i.e. $dim(X/Y) < \infty$. Then, Y is complemented.

D Small perturbations of invertible operators are still invertible

Lemma 20. *Let $A : X \to Y$ be bounded and invertible linear operator. Then, there exists $\epsilon = \epsilon(A)$ so that for each $B : X \to Y$, $\|B\| < \epsilon$, we have that $A + B$ is also invertible.*

Remark: In fact, one may take $\epsilon = \frac{1}{\|A^{-1}\|}$.

References

[1] T. Bühler and D. Salamon, Functional Analysis (Graduate Studies in Mathematics), American Mathematical Society, **191** Providence, RI, 2018.