PROJECT V - MATH 800
DUE MAY 9TH, 2019

(1) (Uniqueness of analytic extensions) Let $f : \Omega \to \mathbf{C}$ be a a holomorphic function and Ω be a domain (i.e. open and connected set). Assume that $\Omega_1 : \Omega \subset \Omega_1$ is a larger domain and $g : \Omega_1 \to \mathbf{C}$ is an analytic extension of f. Show that g is unique, i.e. every other extension should coincide with g on Ω_1.

(2) Let f be a holomorphic function on a domain Ω. Assume that at some point $z_0 \in \Omega$, z_0 real, all derivatives of f are real. That is, $f^{(k)}(z_0) \in \mathbf{R}$. Show that for all $z \in \Omega$, we have the functional identity

$$\overline{f(z)} = f(\bar{z})$$

Hint: First show it for a neighborhood of z_0. Then consider the function $\overline{f(\bar{z})}$. We have used that fact for the Riemann zeta function - we have showed the identity only for $\Re z > 1$ and then, we have it everywhere.

(3) Prove that the function

$$f(z) = \int_0^\infty \frac{e^{-zt}}{1+t^2} dt$$

is holomorphic in $\Re z > 0$ and is continuous in $\Re z \geq 0$.
Hint: The continuity at $\Re z = 0$ does not follow from a general result and it needs to be done explicitly.

(4) Show that

$$f(z) = \prod_{n=1}^\infty (1 - \frac{z}{\sqrt{n}}) e^{(a\frac{z}{\sqrt{n}} + b\frac{z^2}{n})}$$

for appropriately chosen a, b, defines an entire function, with zeros exactly at $\{\sqrt{n}, n = 1, 2, \ldots\}$.
Hint: Use the approach of Lemma 8.2.1, then apply Theorem 8.1.7.

(5) Show that the Riemann zeta function $\zeta(z)$ has holomorphic extension for $\Re z > -1$, with only a simple pole at $z = -1$.
Hint: The basic idea of analytic continuation is: you need a formula in $\Re z > 1$, which involves $\zeta(z)$, so that every other function involved in it is well-defined in $\Re z > -1$. Use the idea in Lemma 16.2.1. More precisely, start by showing the identity, for $z : \Re z > 1$,

$$\zeta(z) - \frac{1}{z-1} - 1 = \sum_{n=1}^\infty \int_n^{n+1} \left(\frac{1}{n^z} - \frac{1}{x^z} - \frac{z}{x^{z+1}} \right) dx. \tag{1}$$

Then, show that the sum on the right defines a holomorphic function in $\Re z > -1$. One can use this procedure to extend $\zeta(z)$ to $\Re z > -k$, for every k. What identity like (1) do you need to make the extension to $\Re z > -2$?