PROJECT II - MATH 800
DUE MARCH 21ST, 2019

(1) Problem 4/p. 94.

(2) Let D be a connected domain, not necessarily simply connected. Assume that there is a sequence of holomorphic on D functions $\{f_j\}$, so that $f_j \rightrightarrows f$, i.e. it converges uniformly on the compact subsets of D to a holomorphic function[1] f. More precisely, this means

$$\forall K \Subset D, \forall \epsilon > 0, \exists N = N(\epsilon, K) : j > N, \sup_{z \in K} |f_j(z) - f(z)| < \epsilon.$$

Prove that

$$f_j' \rightrightarrows f'$$

Hint: Let $K \Subset D$. Choose a closed curve γ inside D, with positive orientation, so that K is in the interior of γ. Then, by the Cauchy integral formula,

$$f_j(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_j(\xi)}{\xi - z} d\xi$$

and hence

$$f_j'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_j(\xi)}{(\xi - z)^2} d\xi.$$

Use now the convergence of f_j on the compact γ to conclude.

(3) Let f be a holomorphic function on a domain D. Let $z_0 \in D : f(z_0) = 0$. Prove that

$$g(z) := \begin{cases} \frac{f(z)}{(z-z_0)} & z \neq z_0 \\ f'(z_0) & z = z_0 \end{cases}$$

is holomorphic in D.

Note that as a consequence, one can always write $f(z) = (z - z_0)g(z)$ with $g \in H(D)$, whenever $f(z_0) = 0$.

Hint: One way to go is to use Morera's theorem. You can also use a theorem like Theorem 2.3.3. For the Morera'a approach, write

$$\int_\gamma g(z)dz = \int_\gamma g(z)dz + \int_{\Gamma_\epsilon} g(z)dz - \int_{\Gamma_\epsilon} g(z)dz$$

where $\Gamma_\epsilon = \{z : |z - z_0| = \epsilon\}$ traced clockwise. By Cauchy theorem, show that

$$\int_\gamma g(z)dz + \int_{\Gamma_\epsilon} g(z)dz = \int_{\gamma \cup \Gamma_\epsilon} g(z)dz = 0$$

and then show that $\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} g(z)dz = 0$.

[1]This is actually always the case, don't need to assume it. That is, if f_j are holomorphic, then its limit f is holomorphic as well

(4) Problem 24/page 97 - Prove that is correct. In fact, show that the radius of convergence is $r' = \min(r, 1)$.

(5) Problem 27/page 97.
Hint: For the case when g has zeros - show first that the function $\frac{f(z)}{g(z)}$ is still well-defined and analytic at the zeros of g. For example, you can argue that it has removable singularities there.

(6) Problem 30/page 98, without the part about improving to a polynomial estimate.

(7) 37/page 99.
Hint: Here, you may want to use an extension of Exercise 2 (which needs to be proved), namely that for each integer k, $f_j^{(k)} \rightrightarrows f^{(k)}$.

(8) 42/page 100.