DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS
MIDTERM MATH 800 - Spring 2012

Your Name: ______________________________

1	(60)	________
2	(100)	________
3	(120)	________
4	(120)	________
Total	(400)	________

(1) (60 points)
Suppose Ω is an open set and $P_0 \subset D(P_0, r) \subset \Omega$. Let $f : \Omega \setminus \{P_0\} \to \mathbf{C}$ is holomorphic, so that for soem constant C and for all $z \in D(P_0, r) \setminus \{P_0\}$,

$$|f(z)| \leq \frac{C}{\sqrt{|z - P_0|}}.$$

Show that P_0 is removable singularity for f.

(2) (100 points)

Is there an analytic function on D, so that $f : D \to D$ with $f(0) = \frac{1}{2}$ and $f'(0) = \frac{3}{4}$. If so, find such an f. Is it unique?

Hint: Apply an appropriate Möbius transformation.

(3) (120 points)
Let $\Omega \subset \mathbf{C}$ be an open and connected set, so that $\overline{D} = \{z : |z| \leq 1\} \subset \Omega$. Let $f : \Omega \to \mathbf{C}$ be a non-constant holomorphic function. Show that if $|f(z)| = 1$ whenever $|z| = 1$, then $f(D) \supset D$.
Hint: The claim follows by establishing that for every $w_0 \in D$, there is $z_0 \in D$, so that $f(z_0) = w_0$. This should be done in a few steps:
(a) Show (by argument principle) that the equation $f(z) = w_0$ has a solution, if $f(z) = 0$ has a solution. Justify the step.
(b) Show that $f(z) = 0$ has a solution (otherwise, apply the maximum modulus principle for both $g(z) = \frac{1}{f(z)}$ and $f(z)$). In the contradiction argument, you must rule out the situation, where f is a constant function in D.

(4) (120 points)

Let $\{a_j\}_{j=1}^{\infty}, a_j \neq 0$ be a sequence of complex numbers, without any point of accumulation.

- Let $\alpha > \frac{1}{2}$ and assume that there is $C > 0$, so that $|a_j| \geq Cj^{\alpha}$. Show that the Weierstrass function $E(z)$
$$E(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right) e^{\frac{z}{a_j}}$$
has $\{a_j\}$ as zeroes. Justify your work.
Hint: Recall our approach for Exercise 10/page 275.
- Assuming only $\alpha > \frac{1}{3}$ and $|a_j| \geq Cj^{\alpha}$, construct Weierstrass function $E(z)$ with zeroes exactly in the set $\{a_j\}$ in the form
$$E(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right) e^{p_j(z)}$$
where $\{p_j\}$ are polynomials of degree **two**. Justify your construction.
Hint: Take $p_j(z) = \frac{z}{a_j} + \beta_j z^2$, with appropriate choice of β_j.