SOLUTION OF SOME PROBLEMS FROM PROJECT I

Problem 14/page 6
We use the method of characteristics for $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$. Write it first as $\frac{\partial u}{\partial x}+\frac{y}{x}\frac{\partial u}{\partial y}=0$. We need to solve

$$\frac{dy}{dx}=\frac{y}{x},$$

which has a solution in the form $y=Cx$. Thus, the general solution is in the form $u(x,y)=f(y/x)$.
Problem 27/page 25
a) Function is NOT piecewise continuous (since $\lim_{x\to 0+}\sin(1/x)$ does not exist.
b), c) Functions are piecewise continuous, but NOT piecewise smooth, since their derivative do not have (one sided) limits as $x\to 0$.
d) Functon is piecewise smooth.
Problem 16/page 59
According to the Parseval's identity ((6), page 56), we have

$$\int_{-\pi}^{\pi} f^2(x)dx=\pi(\sum_{n=1}^{\infty} e^{-2n})=\pi\frac{e^{-2}}{1-e^{-2}}=\frac{\pi}{e^2-1}$$

Problem 15/page 67
Using the results of Example 1, page 63, we get

$$\begin{aligned}
\cosh(ax) &= \frac{e^{ax}+e^{-ax}}{2}= \\
&= \frac{\sinh \pi a}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}(a+in)e^{inx}-\frac{\sinh \pi a}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}(-a+in)e^{inx}= \\
&= \frac{a\sinh \pi a}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}e^{inx}
\end{aligned}$$

Applying the Parseval's identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cosh^2(ax)dx=\frac{a^2(\sinh \pi a)^2}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(a^2+n^2)^2}$$

Computing the integral yields $\int_{-\pi}^{\pi}\cosh^2(ax)dx=\pi+\sinh(2\pi a)/(2a)$, whence

$$\sum_{n=-\infty}^{\infty}\frac{1}{(a^2+n^2)^2}=\frac{\pi}{2a^2(\sinh \pi a)^2}\left[\pi+\frac{\sinh(2\pi a)}{2a}\right].$$

For part b), we take the same approach, this time using

$$\begin{aligned}\sinh(ax) &= \frac{e^{ax}-e^{-ax}}{2} = \\ &= \frac{\sinh \pi a}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}(a+in)e^{inx} + \frac{\sinh \pi a}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}(-a+in)e^{inx} = \\ &= \frac{i\sinh \pi a}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n n}{a^2+n^2}e^{inx},\end{aligned}$$

whence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sinh^2(ax)dx = \frac{(\sinh \pi a)^2}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{n^2}{(a^2+n^2)^2}$$

Since $\int_{-\pi}^{\pi}\sinh^2(ax)dx = \sinh(2\pi a)/(2a) - \pi$, we obtain

$$\sum_{n=-\infty}^{\infty}\frac{n^2}{(a^2+n^2)^2} = \frac{\pi}{2(\sinh \pi a)^2}\left[\frac{\sinh(2\pi a)}{2a} - \pi\right].$$