SOLUTION OF SOME PROBLEMS FROM PROJECT IV

Problem 18/page 25 (out of 3 points)
We have (see (4) on page 218),

$$u = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)) r^n$$

So, the equation $100 - 2\cos(2\theta) = u_r(1, \theta) + 2u(1, \theta)$ reduces to

$$100 - 2\cos(2\theta) = 2a_0 + \sum_{n=1}^{\infty} ((2+n) a_n \cos(n\theta) + (2+n) b_n \sin(n\theta))$$

It follows that $2a_0 = 100$, $a_n = 0, n \neq 2$; $b_n = 0, n \geq 1$ and $-2 = 4a_2$. Thus,

$$u = 50 - \frac{1}{2} r^2 \cos(2\theta).$$

Problem 5/page 236 (out of 3 points)
For the unit disc, the eigenvalues of Δ, with Dirichlet data $u = 0$ on the boundary are $-\alpha_{m,n}^2$, with eigenvectors $\cos(m\theta) J_m(\alpha_{mn} r)$, $\sin(m\theta) J_m(\alpha_{mn} r)$, see Theorem 1 om p. 232. Thus

$$u = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (a_{mn} \cos(m\theta) + b_{mn} \sin(m\theta)) J_m(\alpha_{mn} r).$$

The equation $1 = \Delta u + u$ then looks

$$1 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (a_{mn}(1 - \alpha_{mn}^2) \cos(m\theta) + b_{mn}(1 - \alpha_{mn}^2) \sin(m\theta)) J_m(\alpha_{mn} r)$$

Clearly, $a_{mn} = 0, m \geq 1$, $b_{mn} = 0$ for all m. Thus, we are reduced to

$$1 = \sum_{n=1}^{\infty} a_{0n}(1 - \alpha_{0n}^2) J_0(\alpha_{0n} r)$$

Similar to Example2, page 235, we find

$$a_{0n}(1 - \alpha_{0n}^2) = -\frac{2}{\alpha_{0n}^3 J_1(\alpha_{0n})}.$$

Thus,

$$u(r, \theta) = \sum_{n=1}^{\infty} -\frac{2}{\alpha_{0n}^3 (1 - \alpha_{0n}^2) J_1(\alpha_{0n})} J_0(\alpha_{0n} r)$$

Problem 16/page 408 (out of 4 points)
Note that

$$g(x) = \frac{2x}{(a^2+x^2)^2} = -\left(\frac{1}{a^2+x^2}\right)'$$

So, by the rules for FT for derivatives,

$$\hat{g}(\omega) = -i\omega \widehat{\frac{1}{a^2+x^2}}(\omega).$$

But $\frac{1}{a^2+x^2} = a^{-2}\frac{1}{1+(x/a)^2}$ and recall $\widehat{\frac{1}{1+x^2}}(\omega) = \sqrt{\frac{\pi}{2}}e^{-|\omega|}$. So, denote $h(x) = \frac{1}{1+x^2}$.

$$\widehat{\frac{1}{a^2+x^2}}(\omega) = a^{-2}\widehat{h(x/a)}(\omega) = a^{-1}\hat{h}(a\omega) = a^{-1}\sqrt{\frac{\pi}{2}}e^{-a|\omega|}$$

Finally, taking all into account

$$\hat{g}(\omega) = -\frac{i\omega}{a}\sqrt{\frac{\pi}{2}}e^{-a|\omega|}.$$