SOLUTION OF SOME PROBLEMS FROM PROJECT III

Problem 6/page 185 (out of 4 points)
Using Exercise # 4 as suggested, we have

$$u(x,y) = \sum_{m=0}^{\infty} \cos(\frac{m\pi}{a}x)(A_m \cosh(\frac{m\pi}{a}(b-y)) + B_m \sinh(\frac{m\pi}{a}(b-y)))$$

This satisfies the PDe and the Neumann b.c. at $x = 0, x = a$. Now, the condition $u_y(x,b) = 0$ implies $B_m = 0$. So,

$$u(x,y) = \sum_{m=0}^{\infty} A_m \cos(\frac{m\pi}{a}x) \cosh(\frac{m\pi}{a}(b-y)).$$

It remains to satisfy the last condition, namely $u(x,0) + u_y(x,0) = 0$. We have

$$0 = u(x,0) + u_y(x,0) = \sum_{m=0}^{\infty} A_m \cos(\frac{m\pi}{a}x) \left[\cosh(\frac{m\pi}{a}b) - \frac{m\pi}{a} \sinh(\frac{m\pi}{a}b)\right].$$

This means that for all integers m for which $\cosh(\frac{m\pi}{a}b) - \frac{m\pi}{a}\sinh(\frac{m\pi}{a}b) \neq 0$, we have that $A_m = 0$. For the integers m, for which $\cosh(\frac{m\pi}{a}b) = \frac{m\pi}{a}\sinh(\frac{m\pi}{a}b)$, we have that *an arbitrary* A_m gives us a solution. So, the equation has the solution $u = 0$, and in addition denoting the set of integers (which may be empty!)

$$C_{a,b} = \{m : \cosh(\frac{m\pi}{a}b) = \frac{m\pi}{a}\sinh(\frac{m\pi}{a}b)\}$$

we have other solutions,

$$u = \sum_{m \in C_{a,b}} A_m \cos(\frac{m\pi}{a}x) \cosh(\frac{m\pi}{a}(b-y)).$$

where A_m are arbitrary.
Problem 10/page 206 (out of 3 points)

Using the method of separation of variables $u(r,\theta) = T(t)R(r)$, we obtain

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = const = \frac{T''}{c^2 T}.$$

From the method of this section, see page 201, we obtain

$$R_n(r) = J_0(\lambda_n r), \lambda_n = \frac{\alpha_n}{a}, const = -\lambda_n^2.$$

Thus, the equation to $T : T'' = -c^2\lambda_n^2 T(t)$ resolves to $T_n(t) = const e^{-c^2\lambda_n^2 t}$. Altogether,

$$u = \sum_{n=1}^{\infty} A_n e^{-c^2\lambda_n^2 t} J_0(\lambda_n r).$$

The initial condition at $t = 0$ yields

$$f(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r).$$

This is just a Bessel's expansion, so we obtain (see formula (10) on page 202),

$$A_n = \frac{2}{a^2 J_1^2(\aleph_n)} \int_0^a f(r) J_0(\lambda_n r) r dr.$$

Problem 12/page 216 (out of 3 points)
From the formulas on p. 211 - (12), (13), (14), it is clear that $a_{mn} = 0$ (integration against $\cos(m\theta)$, $m = 0, 1, 2, \dots$) and also $b_{mn} = 0, m \geq 2$ (integration against $\sin(m\theta)$, $m = 2, \dots$). Thus, the only non-zero coefficient is b_{1n}. For it, we have

$$b_{1n} = \frac{2}{J_2^2(\alpha_{1n})} \int_0^1 (1 - r^2) r^2 J_1(\alpha_{1n} r) dr.$$

From formula (15) on page 211, we conclude

$$b_{1n} = \frac{4}{\alpha_{1n}^2 J_2^2(\alpha_{1n})} J_3(\alpha_{1n}).$$