SOLUTION OF SOME PROBLEMS FROM PROJECT II

Problem 12/page 145 (out of 3 points)
We apply the formula (9), p. 140 with $u_1(x) = 100$. Then, it remains to solve

$$\left| \begin{array}{l} u_t = u_{xx}, 0 < x < 1 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 50x(1-x) - 100. \end{array} \right.$$

So, the solution to this PDE is given by

$$u_2(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 t} \sin(n\pi x),$$

where

$$b_n = 2\int_0^1 [50x(1-x) - 100] \sin(n\pi x) dx = 100 \frac{(n^2\pi^2 - 1)((-1)^n - 1)}{n^3\pi^3}.$$

So,

$$u(t,x) = 100 - 200 \sum_{k=0}^{\infty} \frac{((2k+1)^2\pi^2 - 1)}{(2k+1)^3\pi^3} e^{-((2k+1)\pi)^2 t} \sin((2k+1)\pi x),$$

Problem 15/page 153 (out of 3 points)
The separation of variables leads to

$$X''(x) + \mu^2 X(x) = 0, X'(0) = 0, X'(L) = -\kappa X(L).$$

Clearly, $X(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x)$. The condition $X'(0) = 0$ then implies that $C_2 = 0$, so

$$X(x) = C \cos(\mu x).$$

Imposing the other boundary condition leads to the relation

$$\cot(\mu L) = \frac{\mu}{\kappa}.$$

This has a sequence of solutions $\mu_n, n = 1, \ldots,$ $X_n(x) = \cos(\mu_n x)$. Consequently, $T_n(t) = c_n e^{-\mu_n^2 t}$, whence

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\mu_n^2 t} \cos(\mu_n x).$$

Setting this at $t = 0$ yields

$$f(x) = \sum_{n=1}^{\infty} c_n \cos(\mu_n x).$$

Taking into account the orthogonality of $\{\cos(\mu_n x)\}_n$, we obtain, just as in (11) on p. 149,

$$c_n = \frac{1}{\int_0^L \cos(\mu_n x)^2 dx} \int_0^L f(x) \cos(\mu_n x) dx.$$

Problem 12/page 169 (out of 4 points)
We setup the separation of variables in the usual way $u = X(x)Y(y)Z(z)$ and plug in the equation. Dividing by XYZ, we obtain

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0.$$

By boundary conditions $X(0) = X(a) = Y(0) = Y(b) = Z(0) = 0$. Since all three functions depend upon a different variable, we have that

$$\frac{X''}{X} = -\mu^2, \frac{Y''}{Y} = -\nu^2,$$

whence

$$\frac{Z''}{Z} = (\mu^2 + \nu^2).$$

The X and Y equations has been solved already, we obtain

$$\mu = \mu_m = \frac{m\pi}{a}, X_m = \sin(\frac{m\pi}{a} x)$$
$$\nu = \nu_n = \frac{n\pi}{b}, Y_n = \sin(\frac{n\pi}{b} y)$$

And finally

$$\lambda_{mn} = \sqrt{\mu_m^2 + \nu_n^2}, \quad Z(z) = A \cosh(\lambda_{mn} z) + B \sinh(\lambda_{mn} z).$$

Since $Z(0) = 0$, it follows that $A = 0$. As a consequence

$$u(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin(\frac{m\pi}{a} x) \sin(\frac{n\pi}{b} y) \sinh(\lambda_{mn} z)$$

Setting $z = c$ leads to the relation

$$f(x, y) = u(x, y, c) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin(\frac{m\pi}{a} x) \sin(\frac{n\pi}{b} y) \sinh(\lambda_{mn} c).$$

By double Fourier series expansions,

$$A_{mn} = \frac{4}{ab \sinh(\lambda_{mn} c)} \int_0^a \int_0^b f(x, y) \sin(\frac{m\pi}{a} x) \sin(\frac{n\pi}{b} y) dx dy.$$