SOLUTION OF SOME PROBLEMS FROM PROJECT I

Problem 12/page 36
We setup the integrals for the Fourier coefficients as on page 27 (note the a coefficients are zero, since the function is odd) and compute through a computer algebra system (I used Mathematica)

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi^2 x - x^3)dx = 0 \\ a_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi^2 x - x^3)\cos(nx)dx = 0 \\ b_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi^2 x - x^3)\sin(nx)dx = -\frac{12}{n^3}\cos(n\pi) = (-1)^{n+1}\frac{12}{n^3} \end{aligned}$$

The Fourier series is then,

$$12\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin(nx).$$

Problem 17/page 47
Using the formulas in Exercise 4, for $p = \pi$ and at $x = 0$, and noting that the function is continuous at zero, we obtain the following

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2}\dots\right]$$

Rearranging terms yields

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\dots = \frac{\pi^2}{12}.$$

The second part is just playing with the sums

$$A = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots, \quad B = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

From part a), we have that $\frac{\pi^2}{12} = A - \frac{B}{4}$, while

$$B - A = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots = \frac{1}{4}B.$$

From here $B = \frac{4A}{3}$. Plugging this into the other relation yields

$$\frac{\pi^2}{12} = A - \frac{B}{4} = \frac{2A}{3},$$

or $A = \frac{\pi^2}{8}$ as required.
Problem 15 a/page 67

We start with the expansion of e^{ax} , provided in Example 1, page 63.

$$e^{ax} = \frac{\sinh(\pi a)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}(a+in)e^{inx}, -\pi < x < \pi.$$

Consequently, write

$$\frac{e^{ax}}{\sinh(\pi a)} = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}(a+in)e^{inx}, -\pi < x < \pi.$$

Take a derivative in a in the last identity

$$\partial_a \left[\frac{e^{ax}}{\sinh(\pi a)}\right] = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{(n^2-a^2)-2ain}{(a^2+n^2)^2} e^{inx}$$

Applying the Parseval's identity - Theorem 2, page 65,

$$\frac{1}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{|(n^2-a^2)-2ain|^2}{(a^2+n^2)^4} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\partial_a \left[\frac{e^{ax}}{\sinh(\pi a)}\right]\right|^2 dx.$$

Thus, using computer algebra system to compute the integral on the right,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a^2+n^2)^2} = \frac{\pi}{2a^3}\left[\coth[a\pi] + a\pi csch^2[a\pi]\right].$$

Problem 12/page 124

By separation of variables $u = X(x)T(t)$, we obtain

$$T''X + 2kXT' = c^2X''T.$$

which results in

$$\frac{T''}{T} + 2k\frac{T'}{T} = c^2\frac{X''}{X}.$$

These are functions of t on the left and of x on the right, so they are constants. So $\frac{X''}{X} = \mu, X(0) = X(L) = 0$. It follows that $X_n(x) = \sin(n\pi x/L)$, $\mu_n = -(n\pi/L)^2$, $n = 1, 2, \ldots$. The equation for T is

$$T'' + 2kT' + (n\pi c/L)^2T = 0$$

By solving the characteristic equation, we obtain $r = -k \pm \sqrt{k^2 - (n\pi c/L)^2}$. If $k^2 > (n\pi c/L)^2$, we have

$$T_n(t) = e^{-kt}(a_n \cosh(\lambda_n t) + b_n \sinh(\lambda_n t)), \lambda_n = \sqrt{k^2 - (n\pi c/L)^2}$$

If $k^2 < (n\pi c/L)^2$, we have

$$T_n(t) = e^{-kt}(a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)), \lambda_n = \sqrt{(n\pi c/L)^2 - k^2}$$

Thus,

$$\begin{aligned} u(x,t) &= e^{-kt}\sum_{n<\frac{kL}{\pi c}} \sin(n\pi x/L)(a_n\cosh(\lambda_n t)+b_n\sinh(\lambda_n t)) + \\ &+ e^{-kt}\sum_{n>\frac{kL}{\pi c}} \sin(n\pi x/L)(a_n\cos(\lambda_n t)+b_n\sin(\lambda_n t)). \end{aligned}$$

Taking $t=0$ and the specifics of the *sin* expansions, $f(x)=\sum_n a_n\sin(n\pi x/L)$, so

$$a_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)dx.$$

whereas

$$g(x) = u_t(x,0) = \sum_n (\lambda_n b_n - k a_n)\sin(n\pi x/L),$$

whence $\lambda_n b_n - k a_n = \frac{2}{L}\int_0^L g(x)\sin(n\pi x/L)dx$.