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ASYMPTOTIC STABILITY FOR SPECTRALLY STABLE LUGIATO-LEFEVER SOLITONS IN PERIODIC WAVEGUIDES

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ABSTRACT. We consider the Lugiato-Lefever model of optical fibers in the periodic context. Spectrally stable periodic steady states were constructed recently in: L. Delcey-M. Haragus, *Phil. Trans. R. Soc.*, 376, p. 376, (2018), and to appear *Rev. Roumaine Maths. Pures Appl.*, by S. Hakkaev-M. Stanislavova and A. Stefanov, [7].

The spectrum of the linearization around such solitons consists of simple eigenvalues $0, -2\alpha < 0$, while the rest of it is a subset of the vertical line $\{\mu : \Re\mu = -\alpha\}$. Assuming such property abstractly, we show that the linearized operator generates a C_0 semigroup and more importantly, the semigroup obeys (optimal) exponential decay estimates. Our approach is based on the Gearhart-Prüss theorem, where the required resolvent estimates may be of independent interest. These results are applied to the proof of asymptotic stability with phase of the steady states.

1. INTRODUCTION

The Lugiato-Lefever model has been object of intense investigations in the last decade, as it arises as a relevant model in optical fiber. More specifically, it is an envelope models were derived from the Maxwell's equation, [10] to describe the mechanism of pattern formation in the optical field of a cavity filled with Kerr medium, which is then subjected to a radiation field. Related to this, high frequency optical combs generated by micro resonators have been considered in [1, 9, 10, 13, 11], among others. There are also deep experimental studies, [5], which confirm the relevance of this and related models.

There are numerous papers dealing with the model derivation, as well as further reductions to dimensionless variables, [2, 10, 11]. The model equation, considered in [3, 4, 14] is the following

$$\psi_t + i\beta\psi_{xx} + (\gamma + i\delta)\psi - i|\psi|^2\psi = F. \tag{1.1}$$

In this paper, we follow slightly different but equivalent formulation, see for example the derivation in [13] and [12]. This is in the context of the whispering gallery mode resonators. More precisely, we will be studying

$$iu_t + u_{xx} - u + 2|u|^2 u = -i\alpha u - h, t \geq 0, -T \leq x \leq T \tag{1.2}$$

where u is the field envelope (complex-valued) function, t is the normalized time, x is the azimuthal coordinate, while $\alpha > 0$ is the detuning/damping parameter and the normalized pumping strength parameter is $h > 0$. The main interest will be in in stationary/time independent solutions and their stability properties. These are of great interest to the practitioners, especially

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the stable ones and they are often referred to as frequency combs. We remark that mathematically, this model could be done in the periodic (and also in the whole line context), with a definite preference to the periodic case, as the dependence of the period T is an important aspect for the application.

To set the ideas, we are interested in solutions in the form $u(t,x)=\varphi(x)$ and their stability. They satisfy the following time-independent equation

$$-\varphi''+\varphi-2|\varphi|^2\varphi=i\alpha\varphi+h, -T\leq x\leq T \tag{1.3}$$

In several recent works, several different type of such solutions were shown to exist. Many of them turn out to be spectrally unstable, but some of them are stable, which makes them interesting from the point of view of their practical applications. Our results herein address the dynamics close to these spectrally stable solutions.

The main result of this article is about the asymptotic stability of such solutions. In order to state the main result, let us state its linearization around the solution φ. Set $u=\varphi+v, \varphi=\varphi_1+i\varphi_2, v=v_1+iv_2$. After ignoring $O(|v|^2)$ terms, we obtain the following system

$$\begin{pmatrix} -\partial_t v_2 \\ \partial_t v_1 \end{pmatrix} = \begin{pmatrix} -\partial_x^2+1-(6\varphi_1^2+2\varphi_2^2) & -4\varphi_1\varphi_2 \\ -4\varphi_1\varphi_2 & -\partial_x^2+1-(2\varphi_1^2+6\varphi_2^2) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \alpha \begin{pmatrix} v_2 \\ -v_1 \end{pmatrix}$$

In the eigenvalue ansatz, $\begin{pmatrix} v_1(t,x) \\ v_2(t,x) \end{pmatrix} \to e^{\lambda t} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}$, and after rearranging terms, we obtain the eigenvalue problem

$$(\mathcal{J}\mathcal{L}-\alpha)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{1.4}$$

where

$$\mathcal{J}:=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \mathcal{L}:=\begin{pmatrix} -\partial_x^2+1-(6\varphi_1^2+2\varphi_2^2) & -4\varphi_1\varphi_2 \\ -4\varphi_1\varphi_2 & -\partial_x^2+1-(2\varphi_1^2+6\varphi_2^2) \end{pmatrix}. \tag{1.5}$$

Introduce the linearized operator $\mathcal{H}:=\mathcal{J}\mathcal{L}-\alpha$. In the recent papers, [3, 4, 7], spectrally stable solutions of (1.2) were constructed. The common feature of these smooth solutions φ was that the linearized operator $\mathcal{H}$ has its spectrum to the left of the imaginary axes. In fact, $\sigma(\mathcal{H})\cap i\mathbf{R}=\{0\}$, 0 is a simple eigenvalue and the rest of the spectrum is inside $\{\mu:\Re\mu\leq-\alpha\}$. We now take this as an abstract assumption on φ, of which the solitons exhibited in [3, 4, 7] are an example of, see (1.6) below for a detailed description. We aim at showing that such $\mathcal{H}$ generate semigroup with exponential decay and consequently, the solitons are asymptotically stable.

The following is the main result of this paper.

Theorem 1. *Let $\varphi\in H^1[-T,T]$ be a solution of* (1.3). *Assume that the linearized operator $\mathcal{H}$, satisfies $\mathcal{H}\begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix}=0$ and*

$$\sigma(\mathcal{H})\subset\{-2\alpha\}\cup\{0\}\cup\{\Re\lambda=-\alpha\}. \tag{1.6}$$

where $0,-2\alpha$ are simple eigenvalues. Then, $\mathcal{H}$ generates a C_0 semigroup. In addition, when projected away from the zero eigenvalue, $e^{t\mathcal{H}}\mathcal{Q}_0$ has exponential decay, see (3.15) *below.*

Finally, φ it is asymptotically stable soliton. More precisely, for every $\beta:0<\beta<\alpha$, there is an $\epsilon=\epsilon_\beta$ and a constant $C=C_\beta$, so that whenever $\|u_0-\varphi\|_{H^1}<\epsilon$, then the solution u exists globally and it satisfies

$$u(t,x))=\varphi(x-\sigma(t))+v(t,x),$$

where $|\sigma'(t)| \leq \epsilon^{3/2} e^{-\beta t}$, $\|v(t,\cdot)\|_{H^1} \leq C\epsilon e^{-\beta t}$. *In particular, there is* $\sigma_\infty := \lim_{t\to\infty} \sigma(t)$, *so that*

$$\|u(t,\cdot) - \varphi(x-\sigma_\infty)\|_{H^1} \leq C\epsilon e^{-\beta t}.$$

Note that $|\sigma_\infty| \leq \int_0^\infty |\sigma'(t)| dt \leq C_\beta \epsilon^{\frac{3}{2}}$.

Remark: The exponent $\frac{3}{2}$ above is not sharp and it can be replaced by any $\gamma < 2$. Informally, if the initial data u_0 is order $O(\epsilon)$ away from the soliton φ, then the asymptotic phase σ_∞ is roughly order $O(\epsilon^2)$ small.

The plan of the paper is as follows. In Section 2, we start by defining some preliminary notions and definitions and we give some well-known results. In Section 3 contains the main new contribution of the article, namely the exponential estimates for the linearized Lugiato-Lefever semigroup. These are shown through a powerful abstract tool, namely the Gearhart-Prüss theorem, which requires an uniform estimate of the correspondent resolvent, close to the imaginary axes. In Section 4, we apply the exponential estimates to derive the nonlinear asymptotic stability with phase of the solitons, as a solution to the full LL equation.

2. Preliminaries

We use the following Fourier series representations for $f \in L^2[-\pi,\pi]$

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}, \ \ \hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx} dx.$$

so that

$$\|f\|_{H^s[-T,T]} = \left(\sum_k (1+|k|^2)^s |\hat{f}(k)|^2\right)^{1/2}$$

2.1. **Semigroup generation and the Gearhart-Prüss theorem.**

We use the standard definition of C_0 semigroups on a Banach space X, which is that there is a one parameter family of bounded operators $T : [0,\infty) \to B(X)$, so that $T(0) = Id, T(t+s) = T(t)T(s)$ and for each fixed $\varphi \in X$, $\lim_{t\to 0+} \|T(t)\varphi - \varphi\|_X = 0$. Its generator $\mathscr{A}$ is always a closed operator, with a domain $D(\mathscr{A})$, defined through the condition $\lim_{t\to 0+} \frac{T(t)x-x}{t}$ exists in the $\|\cdot\|_X$. Clearly, it is not trivial or automatic to say that any given closed operator generates a C_0 semigroup, but we shall use the following simple sufficient condition, in the case of Hilbert spaces, which is adaptation and consequence of Theorem X. 48, p. 240 and the Lemma on p. 244, [17].

Lemma 1. *Suppose that a closed operator* H_0 *on a Hilbert space* X*, has the property* $\Re\langle H_0 x, x\rangle \leq 0$ *(i.e.* $-H_0$ *is accretive) and in addition, there exists* $\delta > 0$*, so that* $\delta - H_0$ *is invertible. Then,* H_0 *generates a contraction semigroup. If* H *is another operator, so that* $H - H_0$ *is a bounded operator, then* H *generates a* C_0 *semigroup as well.*

2.1.1. *$\mathscr{H}$ generates a C_0 semigroup.*

Let us now use this criteria in order to show that $\mathscr{H}$ generates a C_0 semigroup on the Hilbert space $L^2[-T,T]$. Without loss of generality, we may assume that $T = \pi$. Next, let $\mathscr{L}_0$ denotes the differential part of the self-adjoint operator $\mathscr{L}$. In other words,

$$\mathscr{L}_0 := \begin{pmatrix} -\partial_x^2 + 1 & 0 \\ 0 & -\partial_x^2 + 1 \end{pmatrix}.$$

For $\mathscr{H}_0 := \mathscr{J}\mathscr{L}_0 - \alpha$, we have by integration by parts,

$$\Re\langle \mathscr{H}_0 \vec{\psi}, \vec{\psi}\rangle = -\alpha \|\vec{\psi}\|_{L^2}^2 < 0,$$

since $\alpha > 0$. Also, using the Fourier variables and the calculation,

$$\left(\begin{array}{cc} 1+\alpha & k^2+1 \\ -(k^2+1) & 1+\alpha \end{array}\right)^{-1} = \frac{1}{(k^2+1)^2+(1+\alpha)^2}\left(\begin{array}{cc} 1+\alpha & -(k^2+1) \\ (k^2+1) & 1+\alpha \end{array}\right), \tag{2.1}$$

we can invert $1-\mathcal{H}_0$ as follows

$$(1-\mathcal{H}_0)^{-1} = \left((1+\alpha)^2+(-\partial_x^2+1)^2\right)^{-1}\left(\begin{array}{cc} 1+\alpha & -(-\partial_x^2+1) \\ (-\partial_x^2+1) & 1+\alpha \end{array}\right),$$

which is a bounded operator, basically because the matrix in (2.1) is bounded in k. It follows that $\mathcal{H}_0$ generates a contraction semigroup. Clearly $\mathcal{H}-\mathcal{H}_0 = \mathcal{J}\left(\begin{array}{cc} -(6\varphi_1^2+2\varphi_2^2) & -4\varphi_1\varphi_2 \\ -4\varphi_1\varphi_2 & -(2\varphi_1^2+6\varphi_2^2) \end{array}\right)$, is a bounded operator on $L^2\times L^2$, since $\vec{\varphi}\in H^1\subset L^\infty$. Thus, Lemma 1 applies and we have that $\mathcal{H}$ generates a C_0 semigroup.

Our next task will be to show that $e^{t\mathcal{H}}$ has exponential decay bounds, at least when projected away from its zero eigenvalue. This will require much more effort, but the main tool is by now the classical result, namely the Gearhart-Prüss theorem.

2.1.2. *Gearhart-Prüss theorem.*

Theorem 2 (Gearhart-Prüss)**.** *Let $\mathcal{A}$ generates a C_0 semigroup on a Hilbert space, so that $\sigma(A)\subset\{\lambda:\Re\lambda<0\}$. If*

$$\sup_{\mu\in\mathbf{R}}\|(\mathcal{A}-i\mu)^{-1}\|_{B(H)}<\infty,$$

then there exists $\delta>0$, so that $\|e^{t\mathcal{A}}f\|_H\le C_\delta e^{-\delta t}\|f\|_H$.

As we have mentioned above, $\mathcal{H}$ does not exactly fit the Gearhart-Prüss statement, as it has spectrum on the imaginary axes, namely a simple eigenvalue at zero. This is a frequent occurrence in dynamical system theory, as symmetries of the system manifest themselves each as eigenvalues at zero. Luckily, most of this spectrum is finite dimensional, so one could still apply Theorem 2 by projecting away on a finite co-dimension subspace. In order to do that, we need to look into the relevant object that lets us do that, namely the Riesz projections.

2.2. **Riesz projections.** In this section, we identify the concrete form of the Riesz projection onto the eigenspace corresponding to the zero eigenvalue. More precisely, we are looking to write explicitly the projection onto $Ker[\mathcal{H}] = span[\left(\begin{array}{c}\varphi_1' \\ \varphi_2'\end{array}\right)]$. Define, for $\epsilon<\frac{\alpha}{2}$,

$$\mathcal{P}_0 = \frac{1}{2\pi i}\int_{|\xi|=\epsilon}(\xi-\mathcal{H})^{-1}d\xi$$

and $\mathcal{Q}_0 := Id-\mathcal{P}_0$. Denote the corresponding subspaces $P_0=\mathcal{P}_0[L^2\times L^2], Q_0=\mathcal{P}_0[L^2\times L^2]$.

The operator $\mathcal{P}_0$ is a projection onto one dimensional subspace, so it is a rank-one operator. It is naturally given by

$$\mathcal{P}_0 f = \langle f, e_*\rangle\left(\begin{array}{c}\varphi_1' \\ \varphi_2'\end{array}\right)$$

for some vector e_*.

We now aim to identify e_*. To that end, recall that by our assumption on $\mathcal{H}$, (1.6), $\{-\alpha,\alpha\}\in\sigma(\mathcal{J}\mathcal{L})$. Taking adjoints, we see that $\{-\alpha,\alpha\}\in\sigma(\mathcal{L}\mathcal{J})$. We will show that properly normalized eigenvector e_* for the eigenvalue $-\alpha$ will do the job. Indeed, assume $(\mathcal{L}\mathcal{J}+\alpha)e_*=0$. For any $\mu\neq 0, \mu\in\sigma(\mathcal{H})$, consider the corresponding eigenfunction, $\mathcal{H}f_\mu=\mu f_\mu$. Compute

$$\langle f_\mu, e_*\rangle = \frac{1}{\mu}\langle \mathcal{H}f_\mu, e_*\rangle = \frac{1}{\mu}\langle (\mathcal{J}\mathcal{L}-\alpha)f_\mu, e_*\rangle = -\frac{1}{\mu}\langle f_\mu, (\mathcal{L}\mathcal{J}+\alpha)e_*\rangle = 0.$$

Thus, $e_*\perp Q_0$. On the other hand, $e_*\neq 0$, so it follows that $\langle\begin{pmatrix}\varphi_1'\\ \varphi_2'\end{pmatrix}, e_*\rangle\neq 0$, otherwise e_* will be perpendicular to the whole Hilbert space, and hence $e_*=0$. Thus, the normalization of e_* is exactly so that $\langle\begin{pmatrix}\varphi_1'\\ \varphi_2'\end{pmatrix}, e_*\rangle = 1$. This determines e_* uniquely and this is the formula that we use henceforth. In fact, it is easy to see that $\eta:=\mathcal{J}e_*$ is the unique (when properly normalized) eigenvector of $\mathcal{H}\eta=-\alpha\eta$. So, we have proved the following lemma.

Lemma 2. *The projection $\mathcal{P}_0$ onto the eigenspace spanned by $\vec{\varphi}'$ is given by the formula $\mathcal{P}_0 f = \langle f, e_*\rangle\vec{\varphi}'$, where $\mathcal{J}e_*$ is the unique eigenfunction of $\mathcal{H}$ at $-\alpha$, normalized with $\langle\vec{\varphi}', e_*\rangle=1$.*

Next, we present a lemma, which provides an useful decomposition, if one is close to the soliton, with modified radiation term, which is kept in the subspace Q_0.

Lemma 3. *There exists $\epsilon_0>0$, so that whenever $\vec{u}\in H^1(\mathbf{R})\times H^1(\mathbf{R}) : \|\vec{u}-\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\|_{H^1}<\epsilon_0$, then there exists a pair $(\sigma, v)\in\mathbf{R}\times Q_0$, so that $\sigma=\sigma(\vec{u}), \vec{v}=v(\vec{u})$,*

$$\vec{u}=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}(\cdot-\sigma)+\vec{v}, \mathcal{Q}_0 v = v.$$

This is a well-known result, which is by now classical in the asymptotic stability literature. For a proof, in a slightly more general framework, we refer the reader to Lemma 2.2 in [8]. Henceforth, we also adopt the notation $\vec{\varphi}_\sigma(x):=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}(x-\sigma)$.

3. Exponential decay estimates for $e^{t\mathcal{H}}$

We start by the observation that the operator $\mathcal{H}$ generates a C_0 semigroup on $L^2[-T,T]$, as established in Section 2.1.1. We now show appropriate exponential decay estimates for the semigroup $e^{t\mathcal{H}}$ generated by $\mathcal{H}$. We use the Gearhart-Prüss theorem, which calls for uniform resolvent bounds, with respect to the spectral parameter on the imaginary axes. In fact, the difficult part of this estimate is to look for sufficiently large μ, while the control on compact intervals is a softer issue, which essentially reduces to knowing that spectrum does not appear on $i\mathbf{R}$, an assumption in Theorem 1. See the proof of Corollary 1 below.

We have a more general result, which may be of independent interest. Note that estimates of similar flavor (but for the case of the whole line, with all essential spectrum) have been done in the past, see for example [6] and the Ph.D. thesis of the first author [18]. Note that the argument here seems more straightforward as it completely avoids the Birman-Schwinger technology (and the associated compactness considerations). Instead, we precisely identify the problematic spectral parameters and we keep the necessary inversions in the Neumann realm.

Proposition 1. *Let*

$$\mathcal{M}:=\mathcal{J}\begin{pmatrix}-\partial_x^2+1+V_1 & W\\ W & -\partial_x^2+1+V_2\end{pmatrix}=\mathcal{J}L,$$

where V_1, W, V_2 are real-valued and belong to $H^1[-T,T]$. Note that L is a self-adjoint operator, when taken with the domain $H^2[-T,T]\times H^2[-T,T]$.

Then, for every $\delta>0$, there exists N_δ, so that whenever $|\mu|>N_\delta$, $\delta+i\mu\in\rho(\mathcal{M})$. More importantly, we have the estimate

$$\sup_{\mu:|\mu|>N_\delta}\|(\mathcal{M}-(\delta+i\mu)Id)^{-1}\|_{L^2\to L^2}\leq C\delta^{-1}.$$

Remark: The assumptions $V_1, W, V_2\in H^1[-T,T]$ may be weakened to the following

$$\lim_k(|\hat{V}_1(k)|+|\hat{V}_2(k)|+|\hat{W}(k)|)=0.$$

Proof. (Proposition 1) Without loss of generality, we will take $T=\pi$. We need to invert $\mathcal{M}-(\delta+i\mu)=\mathcal{J}(L+(\delta+i\mu)\mathcal{J})$, so we need to provide bounds for the inverse of $L+(\delta+i\mu)\mathcal{J}$. Without loss of generality $\mu>0$, since the other case is symmetric to this one. Thus, we set up the equation

$$(L+(\delta+i\mu)\mathcal{J})\begin{pmatrix} f\\ g\end{pmatrix}=\begin{pmatrix} F\\ G\end{pmatrix}, \tag{3.1}$$

and we need to show the estimate

$$\|\begin{pmatrix} f\\ g\end{pmatrix}\|_{L^2}\lesssim\delta^{-1}\|\begin{pmatrix} F\\ G\end{pmatrix}\|_{L^2}. \tag{3.2}$$

Ignoring for a second the potentials V_1, W, V_2, we see that on the k^{th} Fourier mode, we have the formula (in Fourier multiplier form)

$$\begin{pmatrix} k^2+1 & \delta+i\mu\\ -(\delta+i\mu) & k^2+1\end{pmatrix}^{-1}=\frac{1}{(k^2+1)^2-\mu^2+\delta^2+2i\mu\delta}\begin{pmatrix} -k^2-1 & -(\delta+i\mu)\\ (\delta+i\mu) & -k^2-1\end{pmatrix}. \tag{3.3}$$

Clearly, since $\mu>>1, 0<\delta<<1$, this Fourier multiplier has lots of decay in μ except when $|(k^2+1)^2+\delta^2-\mu^2|\leq\mu$. We claim that if $\mu>100$, there is at most one integer $k_0=k_0(\mu)$, so that

$$|(k_0^2+1)^2+\delta^2-\mu^2|\leq\mu$$

but for all $k\neq\pm k_0(\mu)$, we have

$$|(k^2+1)^2+\delta^2-\mu^2|\geq\frac{1}{10}(\max(|k|^2,\mu))^{\frac{3}{2}}. \tag{3.4}$$

Indeed, the inequality $|(k^2+1)^2+\delta^2-\mu^2|\leq\mu$ leads to $|k^2-\mu|\leq 1$, which then implies $|k-\sqrt{\mu}|\leq\frac{1}{\sqrt{\mu}}$. If now $\mu>100$, clearly, this inequality identifies at most one integer $k_0(\mu)$, namely $k_0(\mu)=]\sqrt{\mu}[$, the closest integer to $\sqrt{\mu}$. Furthermore, if $k\neq k_0(\mu)$, by considering the two cases $k\geq k_0(\mu)+1$ or $k\leq k_0(\mu)-1$ separately, we can show (3.4). Thus, we have shown the following

Lemma 4. *For every $\mu>100$, there exists at most one integer $k_0=k_0(\mu)>0$ (which is necessarily $k_0(\mu)=]\sqrt{\mu}[$), so that*

$$|(k_0^2+1)^2+\delta^2-\mu^2|\leq\mu$$

and for all $k\neq\pm k_0(\mu)$, there is (3.4). *Note that if $k_0(\mu)$ does not exist, then* (3.4) *holds for all k.*

Let us now write equations for the Fourier coefficients $\hat{f}(k), \hat{g}(k)$. Clearly the problematic terms will occur at $\pm k_0(\mu)$. We will, with a slight abuse of notations, not distinguish between an

$L^2[-\pi,\pi]$ function and its (sequence of) Fourier coefficients. In order to introduce convenient notations let $\hat{z}(k) := \begin{pmatrix} \hat{f}(k) \\ \hat{g}(k) \end{pmatrix}$, $z = \sum_{k=-\infty}^{\infty} \hat{z}(k) e^{ikx}$, and

$$\tilde{z} := \sum_{k \neq \pm k_0(\mu)} \hat{z}(k) e^{ikx},$$

Writing the equations for $\hat{z}(k), k \neq k_0(\mu)$, we obtain from (3.1),

$$\hat{z}(k) + \begin{pmatrix} k^2+1 & \delta+i\mu \\ -(\delta+i\mu) & k^2+1 \end{pmatrix}^{-1} \begin{pmatrix} \widehat{V_1 f}(k) + \widehat{Wg}(k) \\ \widehat{Wf}(k) + \widehat{V_2 g}(k) \end{pmatrix} = \begin{pmatrix} k^2+1 & \delta+i\mu \\ -(\delta+i\mu) & k^2+1 \end{pmatrix}^{-1} \begin{pmatrix} \hat{F}(k) \\ \hat{G}(k) \end{pmatrix}$$

Denote $\mathscr{A}_k := \begin{pmatrix} k^2+1 & \delta+i\mu \\ -(\delta+i\mu) & k^2+1 \end{pmatrix}^{-1}$ and observe that due to (3.3) and Lemma 4 (and more concretely (3.4)), we have the bound,

$$\|\mathscr{A}_k\|_{L^2 \to L^2} \leq C(\max(|k|^2, \mu))^{-\frac{1}{2}}, \ \ k \neq \pm k_0(\mu), \tag{3.5}$$

We can rewrite the last equations, after summation in $k \neq \pm k_0(\mu)$, in the following schematics form

$$\tilde{z} + \tilde{\mathscr{A}} z = \mathscr{A} \begin{pmatrix} F \\ G \end{pmatrix} \tag{3.6}$$

where

$$\tilde{\mathscr{A}} z := \sum_{k \neq \pm k_0(\mu)} \mathscr{A}_k \begin{pmatrix} \widehat{V_1 f}(k) + \widehat{Wg}(k) \\ \widehat{Wf}(k) + \widehat{V_2 g}(k) \end{pmatrix}, \ \mathscr{A} \begin{pmatrix} F \\ G \end{pmatrix} = \sum_{k \neq \pm k_0(\mu)} \mathscr{A}_k \begin{pmatrix} \hat{F}(k) \\ \hat{G}(k) \end{pmatrix}.$$

An important information to retain from the representation (3.6) are the bounds,

$$\|\tilde{\mathscr{A}}\|_{L^2 \to L^2} \leq C_{V_1,V_2,W} |\mu|^{-\frac{1}{2}}, \tag{3.7}$$

$$\|\mathscr{A}\|_{L^2 \to L^2} \leq C_{V_1,V_2,W} |\mu|^{-\frac{1}{2}}, \tag{3.8}$$

which follows as a direct consequence of (3.5). Proceeding with the equation for $\hat{z}(k), k = k_0(\mu)$, we obtain from (3.1),

$$\begin{pmatrix} k_0^2 + 1 + \frac{1}{2\pi}\int_{-\pi}^{\pi} V_1(x)dx & \delta + i\mu + \frac{1}{2\pi}\int_{-\pi}^{\pi} W(x)dx \\ -(\delta+i\mu) + \frac{1}{2\pi}\int_{-\pi}^{\pi} W(x)dx & k_0^2 + 1 + \frac{1}{2\pi}\int_{-\pi}^{\pi} V_2(x)dx \end{pmatrix} \hat{z}(k_0) + \mathscr{B}_{k_0} z = \begin{pmatrix} \hat{F}(k_0) \\ \hat{G}(k_0) \end{pmatrix}, \tag{3.9}$$

where

$$\begin{aligned} \mathscr{B}_{k_0} z &= \sum_{k \neq k_0(\mu)} \begin{pmatrix} \hat{V}_1(k_0-k) & \hat{W}(k_0-k) \\ \hat{W}(k_0-k) & \hat{V}_2(k_0-k) \end{pmatrix} \hat{z}(k) = \\ &= \sum_{k \neq \pm k_0(\mu)} \begin{pmatrix} \hat{V}_1(k_0-k) & \hat{W}(k_0-k) \\ \hat{W}(k_0-k) & \hat{V}_2(k_0-k) \end{pmatrix} \hat{z}(k) + \begin{pmatrix} \hat{V}_1(-2k_0) & \hat{W}(-2k_0) \\ \hat{W}(-2k_0) & \hat{V}_2(-2k_0) \end{pmatrix} \hat{z}(-k_0) = \\ &= \tilde{B}_{k_0} \tilde{z} + \mathscr{C}_{k_0} \hat{z}(-k_0). \end{aligned}$$

Note that due to the assumption that $V_1, V_2, W \in H^1$, we have the following decay for its Fourier coefficients $|\hat{V}_1(n)| + |\hat{V}_2(n)| + |\hat{W}(n)| \leq C(1+|n|)^{-1}$, whence

$$\|\tilde{B}_{k_0}\|_{L^2 \to \mathscr{C}^2} \leq C, \ \ \|\mathscr{C}_{k_0}\|_{\mathscr{C}^2 \to \mathscr{C}^2} \leq C(1+|k_0|)^{-1} \sim C\mu^{-1/2}. \tag{3.10}$$

Since we are at the critical case $k = k_0(\mu)$, we use the trivial bound for the determinant, coming from its imaginary part, namely

$$|\det(\begin{pmatrix} k_0^2+1+\frac{1}{2\pi}\int_{-\pi}^{\pi} V_1(x)dx & \delta+i\mu+\frac{1}{2\pi}\int_{-\pi}^{\pi} W(x)dx \\ -(\delta+i\mu)+\frac{1}{2\pi}\int_{-\pi}^{\pi} W(x)dx & k_0^2+1+\frac{1}{2\pi}\int_{-\pi}^{\pi} V_2(x)dx \end{pmatrix}| \geq c\delta\mu \sim c\delta k_0^2(\mu).$$

In this calculcation, it is important to note that the $O(1)$ terms, involving $\int_{-\pi}^{\pi} W(x)dx$ in the imaginary parts cancel out. It follows from (3.3), that

$$\left\| \begin{pmatrix} k_0^2+1+\frac{1}{2\pi}\int_{-\pi}^{\pi} V_1(x)dx & \delta+i\mu+\frac{1}{2\pi}\int_{-\pi}^{\pi} W(x)dx \\ -(\delta+i\mu)+\frac{1}{2\pi}\int_{-\pi}^{\pi} W(x)dx & k_0^2+1+\frac{1}{2\pi}\int_{-\pi}^{\pi} V_2(x)dx \end{pmatrix}^{-1} \right\|_{\mathbf{R}^2\to\mathbf{R}^2} \leq C\delta^{-1}. \tag{3.11}$$

Taking into account (3.9), (3.11), and (3.10), we can rewrite (3.9) in the schematic form

$$\hat{z}(k_0)+O(1)\tilde{z}+O(\mu^{-1/2})\hat{z}(-k_0) = O(1)\begin{pmatrix} \hat{F}(k_0) \\ \hat{G}(k_0) \end{pmatrix}, \tag{3.12}$$

where $O(1)$ represents an $B(L^2)$ operator with norm no bigger than $C\delta^{-1}$ and $\|O(\mu^{-1/2})\|_{\mathscr{C}\to\mathscr{C}} \leq C\delta^{-1}\mu^{-1/2}$. Similar calculation shows that the same form can be deduced for the other problematic Fourier mode $\hat{z}(-k_0)$, namely

$$\hat{z}(-k_0)+O(1)\tilde{z}+O(\mu^{-1/2})\hat{z}(k_0) = O(1)\begin{pmatrix} \hat{F}(-k_0) \\ \hat{G}(-k_0) \end{pmatrix}, \tag{3.13}$$

Furthermore, we can input the form (3.6) in the formulas (3.12) and (3.13). Taking into account (3.7), (3.8), together with (3.6), (3.12) and (3.13), this leads us to the general relation,

$$z+O(\mu^{-1/2})z = O(1)\begin{pmatrix} F \\ G \end{pmatrix}. \tag{3.14}$$

The form (3.14) implies that by Neumann's criteria, for μ large enough (depending on various implicit constants involving the potentials V_1, V_2, W), $Id+O(\mu^{-1/2})$ will be invertible and therefore

$$\|z\|_{L^2} \leq \|(Id+O(\mu^{-1/2}))^{-1}O(1)\begin{pmatrix} F \\ G \end{pmatrix}\|_{L^2} \leq C\delta^{-1}\|\begin{pmatrix} F \\ G \end{pmatrix}\|_{L^2}$$

This is of course nothing but (3.2), so Proposition 1 is proved in full. □

Applying the Proposition 1 to the operator $\mathscr{H}$ and taking into account the Gearhart-Prüss result, we obtain

Corollary 1. *Let $\mathscr{H}$ satisfy the assumption* (1.6). *Then, the semigroup $e^{t\mathscr{H}}$ has growth rate $\omega_0(\mathscr{H}) \leq -\alpha$. More precisely, for every $\delta > 0$, there is C_δ, so that*

$$\|e^{t\mathscr{H}}\mathscr{Q}_0 f\|_{L^2[-T,T]} \leq C_\delta e^{-(\alpha-\delta)t}\|f\|_{L^2[-T,T]}. \tag{3.15}$$

Proof. Recall that $\mathscr{H} = \mathscr{J}\mathscr{L}-\alpha$. Thus, it suffices to prove that the growth rate of $\mathscr{J}\mathscr{L}\mathscr{Q}_0$ is at most zero. Note that $\mathscr{J}\mathscr{L}$ satisfies the assumptions of Proposition 1. It follows that for all $\delta > 0$, and for all large enough μ, $\sup_{|\mu|>N_\delta}\|(\mathscr{J}\mathscr{L}-(\delta+i\mu))^{-1}\|_{B(L^2)} < \infty$. On the other hand, by the assumption (1.6), there is $\sup_{|\mu|<N_\delta}\|(\mathscr{J}\mathscr{L}-(\delta+i\mu))^{-1}\|_{B(L^2)} \leq C_\delta$. This is because $z \to (\mathscr{J}\mathscr{L}-z)^{-1}$ is an analytic function on $\rho(\mathscr{H})$, and as such, it is bounded on any compact subset of $\rho(\mathscr{H})$. Thus,

$$\sup_{\mu\in\mathbf{R}}\|(\mathscr{J}\mathscr{L}-(\delta+i\mu))^{-1}\|_{B(L^2)} \leq M_\delta$$

Restricting on the Hilbert subspace $\mathcal{Q}_0 L^2$, we obtain

$$\sup_{\mu\in\mathbf{R}} \|(\mathcal{J}\mathcal{L}\mathcal{Q}_0 - (\delta + i\mu))^{-1}\|_{B(L^2)} \leq M_\delta \tag{3.16}$$

Now, $\sigma(\mathcal{J}\mathcal{L}\mathcal{Q}_0) \subset \{\lambda : \Re\lambda \leq 0\}$ and this together with (3.16) allows us to apply the Gearhart-Prüss theorem. It implies that there exists $\delta_0 > 0$,

$$\|e^{t(\mathcal{J}\mathcal{L}\mathcal{Q}_0-\delta)} f\|_{L^2[-T,T]} \leq C_\delta e^{-\delta_0 t}\|f\|_{L^2[-T,T]}.$$

It follows that for every $\delta > 0$, there is C_δ,

$$\|e^{t\mathcal{J}\mathcal{L}}\mathcal{Q}_0 f\|_{L^2[-T,T]} = \|e^{t\mathcal{J}\mathcal{L}\mathcal{Q}_0} f\|_{L^2[-T,T]} \leq C_\delta e^{(\delta-\delta_0)t}\|f\|_{L^2[-T,T]} \leq C_\delta e^{\delta t}\|f\|_{L^2[-T,T]},$$

as required. □

We now provide an extension of the exponential decay estimates to the case of Sobolev spaces $H^s[-T,T]$.

Lemma 5. *Let $\mathcal{H}$ satisfiy the assumption* (1.6) *and $s \in (0,2)$. Then, for every $\delta > 0$, there exists C_δ, so that*

$$\|e^{t\mathcal{H}}\mathcal{Q}_0 \vec{f}\|_{H^s[-T,T]\times H^s[-T,T]} \leq C e^{-(\alpha-\delta)t}\|\vec{f}\|_{H^s[-T,T]\times H^s[-T,T]}. \tag{3.17}$$

Proof. The case $s = 0$ has been handled already, see Corollary 1. The general case follows by an interpolation argument as follows. Recall that $\mathcal{L}$ is a self-adjoint operator with a domain $D(\mathcal{L}) = H^2[-T,T] \times H^2[-T,T]$. Moreover, it is clear that $\mathcal{L}$ is bounded from below. This can be seen by taking $\gamma >> 1$ and considering the quadratic form for $\mathcal{L}+\gamma$. Estimating in a straightforward manner

$$\langle(\mathcal{L}+\gamma)\vec{g},\vec{g}\rangle \geq \|\vec{g}\|^2_{\dot{H}^1[-T,T]\times\dot{H}^1[-T,T]} + \gamma^2\|\vec{g}\|^2_{L^2} - C_{\varphi_1,\varphi_2}\gamma\|g\|^2_{L^2} \geq \|\vec{g}\|^2_{H^1[-T,T]\times H^1[-T,T]}$$

Thus, for such large γ, one may define $(\mathcal{L}+\gamma)^{\frac{s}{2}}$, with domain $D((\mathcal{L}+\gamma)^{\frac{s}{2}}) = H^s \times H^s$ and so that

$$\|\vec{g}\|_{H^s\times H^s} \sim \|(\mathcal{L}+\gamma)^{\frac{s}{2}}\vec{g}\|_{L^2}$$

Thus, the estimate (3.17) will follow, by complex interpolation, from the estimates

$$\|(\mathcal{L}+\gamma)^{a+i\mu} e^{t\mathcal{H}}\mathcal{Q}_0\vec{f}\|_{L^2\times L^2} \leq C e^{-(\alpha-\delta)t}\|\vec{f}\|_{H^{2a}[-T,T]\times H^{2a}[-T,T]}, \quad a = 0,1 \tag{3.18}$$

Here, one needs to be able to make sense of the complex powers - this is indeed the case, as a consequence of the Stone's theorem, since the operator $\mathcal{L}+\gamma > 0$. In any case, if (3.18) is verified, it implies that

$$\|e^{t\mathcal{H}}\mathcal{Q}_0\vec{f}\|_{H^s\times H^s} \sim \|(\mathcal{L}+\gamma)^{\frac{s}{2}} e^{t\mathcal{H}}\mathcal{Q}_0\vec{f}\|_{L^2\times L^2} \leq C e^{-(\alpha-\delta)t}\|\vec{f}\|_{H^s[-T,T]\times H^s[-T,T]}, s \in (0,1).$$

So, it remains to check (3.18). The bound for $a = 0$ is obvious from (3.15) and the fact that the operators $(\mathcal{L}+\gamma)^{i\mu}, \mu \neq 0$ are unitary on $L^2 \times L^2$ by Stone's theorem.

For $a = 1$, we have by Stone's theorem $\|(\mathcal{L}+\gamma)^{1+i\mu}e^{t\mathcal{H}}\mathcal{Q}_0\vec{f}\|_{L^2\times L^2} = \|(\mathcal{L}+\gamma)e^{t\mathcal{H}}\mathcal{Q}_0\vec{f}\|_{L^2\times L^2}$. Then ,we have

$$\begin{aligned} &\|(\mathcal{L}+\gamma)e^{t\mathcal{H}}\mathcal{Q}_0\vec{f}\|_{L^2\times L^2} = \|(\mathcal{J}\mathcal{L} - \alpha + \mathcal{J}\gamma + \alpha)e^{t\mathcal{H}}\mathcal{Q}_0\vec{f}\|_{L^2\times L^2} \leq \|\mathcal{H}e^{t\mathcal{H}}\mathcal{Q}_0\vec{f}\|_{L^2\times L^2} + \\ + \quad &C_{\alpha,\gamma}\|e^{t\mathcal{H}}\mathcal{Q}_0\vec{f}\|_{L^2\times L^2} \leq C e^{-(\alpha-\delta)t}(\|\mathcal{H}\vec{f}\|_{L^2\times L^2} + \|\vec{f}\|_{L^2\times L^2}) \leq C e^{-(\alpha-\delta)t}\|f\|_{H^2}. \end{aligned}$$

This finishes the proof of Lemma 5. □

4. ASYMPTOTIC STABILITY: PROOF OF THEOREM 1

We first set up the perturbation equation. We follow the standard construction of the perturbation ansatz, see for example [8].

4.1. **The equation for the perturbation.** For a solution $u = u_1 + iu_2$ of (1.2), and under the *a priori* assumption for smallness, that is $\|\vec{u}(t) - \vec{\varphi}\|_{H^1_x} << 1$, we use the ansatz guaranteed by Lemma 3,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \varphi_1(\cdot - \sigma(t)) \\ \varphi_2(\cdot - \sigma(t)) \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \ \mathcal{Q}_0 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

We obtain the system

$$-\mathcal{J}\partial_t \vec{v} - (\mathcal{L}_\sigma + \alpha\mathcal{J})\vec{v} - \sigma'(t)\mathcal{J}\vec{\varphi'}_\sigma = \begin{pmatrix} N_1(\vec{v}) \\ N_2(\vec{v}) \end{pmatrix}, \tag{4.1}$$

where $\mathcal{L}_\sigma$ is the self-adjoint operator displayed in (1.5), (with $\vec{\varphi}$ replaced by $\vec{\varphi}_\sigma$) and nonlinearity given by

$$\begin{aligned} N_1(\vec{v}) &= -2\left[(\varphi_{1,\sigma} + v_1)^2 + (\varphi_{2,\sigma} + v_2)^2\right](\varphi_{1,\sigma} + v_1)) + \\ &+ 2\left((\varphi_{1,\sigma}^2 + \varphi_{2,\sigma}^2)\varphi_{1,\sigma} + (6\varphi_{1,\sigma}^2 + 2\varphi_{2,\sigma}^2)v_1 + 4\varphi_{1,\sigma}\varphi_{2,\sigma}v_2\right) \\ N_2(\vec{v}) &= -2\left[(\varphi_{1,\sigma} + v_1)^2 + (\varphi_{2,\sigma} + v_2)^2\right](\varphi_{2,\sigma} + v_2)) + \\ &+ 2\left((\varphi_{1,\sigma}^2 + \varphi_{2,\sigma}^2)\varphi_{2,\sigma} + 4\varphi_{1,\sigma}\varphi_{2,\sigma}v_1 + (2\varphi_{1,\sigma}^2 + 6\varphi_{2,\sigma}^2)v_2\right) \end{aligned}$$

Multiplying by $\mathcal{J}$, we arrive at

$$\partial_t \vec{v} - (\mathcal{J}\mathcal{L}_\sigma - \alpha)\vec{v} + \sigma'(t)\vec{\varphi'}_\sigma = \mathcal{J}\begin{pmatrix} N_1(\vec{v}) \\ N_2(\vec{v}) \end{pmatrix}, \tag{4.2}$$

Note that the system (4.2) is still not very good for our purposes, for example L_σ is still a time dependent object at this point.

We have reduced matters to

$$\partial_t \vec{v} - (\mathcal{J}\mathcal{L} - \alpha)\vec{v} + \sigma'(t)\vec{\varphi'}_\sigma = \mathcal{J}\begin{pmatrix} N_1(\vec{v}) \\ N_2(\vec{v}) \end{pmatrix} + \mathcal{J}(\mathcal{L}_\sigma - \mathcal{L})\vec{v}. \tag{4.3}$$

Recall now that Lemma 3 provides us with a decomposition, subordinated to the functional calculus of the operator $\mathcal{H} = \mathcal{J}\mathcal{L} - \alpha$. More precisely, $\vec{v}$ belongs to the subspace Q_0, that is $\mathcal{Q}_0\vec{v} = \vec{v}$, while $\mathcal{P}_0\vec{v} = 0$. At the same time $\mathcal{P}_0\vec{\varphi'} = \vec{\varphi'}$, while $\mathcal{Q}_0\vec{\varphi'} = 0$. Applying the two projections to (4.3), we obtain the system of a coupled PDE and ODE,

$$\begin{cases} \partial_t \vec{v} - \mathcal{H}\vec{v} = \mathcal{Q}_0[\mathcal{J}\begin{pmatrix} N_1(\vec{v}) \\ N_2(\vec{v}) \end{pmatrix} + \mathcal{J}(\mathcal{L}_\sigma - \mathcal{L})\vec{v})] - \sigma'(t)\mathcal{Q}_0[\vec{\varphi'}_\sigma] \\ \sigma'(t)\langle\vec{\varphi'}_\sigma, e_*\rangle = \langle\mathcal{J}\begin{pmatrix} N_1(\vec{v}) \\ N_2(\vec{v}) \end{pmatrix} + \mathcal{J}(\mathcal{L}_\sigma - \mathcal{L})\vec{v}, e_*\rangle. \end{cases} \tag{4.4}$$

Here, we have used that $\mathcal{P}_0\mathcal{H} = \mathcal{H}\mathcal{P}_0, \mathcal{Q}_0\mathcal{H} = \mathcal{H}\mathcal{Q}_0$. Our goal is to show exponential time decay and smallness for both $\|\vec{v}(t,\cdot)\|_{H^1}$ and $|\sigma'(t)|$ and just smallness for $|\sigma(t)|$. The last would follow from smallness/integrability for $|\sigma'(t)|$, since $\sigma(0) = 0$, as we start close to the soliton.

Let us informally analyze the terms appearing in (4.4). We have put the higher order/small terms on the right hand side of (4.4). Indeed, $N_j(\vec{v}), j = 1,2$ are quadratic (or higher) in $\vec{v}$, while the terms $(\mathcal{L}_\sigma - \mathcal{L})\vec{v}$ are in the form $O(\sigma v)$ - that is, they are linear in v, but they contain a small factor, proportional to σ. Finally, $\sigma'(t)$ is multiplied by $\langle\vec{\varphi'}_\sigma, e_*\rangle$ on the left, but note that this is close to $\langle \vec{\varphi}', e_* \rangle = 1$, up to order $O(\sigma)$, so $\langle \vec{\varphi}'_\sigma, e_* \rangle = 1 + O(\sigma)$. This explains the decay/smallness properties of $\sigma'(t)$.

For the $\vec{v}$ equation, note that it is driven by $\mathcal{H}$ on the stable subspace Q_0 and as such, the semigroup operator has exponential decay, according to Corollary 1, see also (3.15). The first two terms on the right are similar to the one discussed for $\sigma'(t)$, while the last term $\mathcal{Q}_0[\vec{\varphi}'_\sigma]$ is close to $\mathcal{Q}_0[\vec{\varphi}'] = 0$. This informal analysis convinces us that the nonlinear problem is well-behaved. In the next section, we present the details of the formal proof.

4.2. **Nonlinear stability of the wave φ.** We start by noting that the ODE/PDE system is supplied by an initial condition, $\vec{v}(0) = \vec{v}_0 \in Q_0 \subset H^1$, which is small enough. This corresponds to the fact that we start close to the solitary wave: $\|u_0 - \varphi\|_{H^1} << 1$. Also, we set $\sigma(0) = 0$. Write

$$\vec{v}(t) = e^{t\mathcal{H}} \vec{v}_0 + \int_0^t e^{(t-s)\mathcal{H}} \mathcal{Q}_0[\mathcal{J} \begin{pmatrix} N_1(\vec{v}(s)) \\ N_2(\vec{v}(s)) \end{pmatrix} + \mathcal{J}(\mathcal{L}_\sigma - \mathcal{L})\vec{v}(s))] - \sigma'(s)\mathcal{Q}_0[\vec{\varphi}'_\sigma] ds \tag{4.5}$$

$$\sigma'(t) = \frac{1}{\langle \vec{\varphi}'_\sigma, e_* \rangle} \langle \mathcal{J} \begin{pmatrix} N_1(\vec{v}) \\ N_2(\vec{v}) \end{pmatrix} + \mathcal{J}(\mathcal{L}_\sigma - \mathcal{L})\vec{v}, e_* \rangle. \tag{4.6}$$

Before we set the nonlinear persistence argument, let us derive some estimates on the nonlinear terms. For the nonlinear term, as it consists of quadratic and cubic terms of $\vec{v}$,

$$\|\mathcal{J} \begin{pmatrix} N_1(\vec{v}(s)) \\ N_2(\vec{v}(s)) \end{pmatrix}\|_{H^1 \times H^1} \le C\|v(s)\|_{H^1}^2 (\|v(s)\|_{H^1} + \|\vec{\varphi}\|_{H^1}),$$

where we have used that H^1 is an algebra. Next,

$$\|(\mathcal{L}_\sigma - \mathcal{L})\vec{v}(s)\|_{H^1 \times H^1} \le C\|\vec{\varphi}_\sigma - \vec{\varphi}\|_{H^1}\|\vec{v}(s)\|_{H^1} \le C|\sigma(s)|\|\vec{\varphi}\|_{H^2}\|\vec{v}(s)\|_{H^1}.$$

Next, $\mathcal{Q}_0[\vec{\varphi}'_\sigma] = \vec{\varphi}'_\sigma - \langle \vec{\varphi}'_\sigma, e_* \rangle \vec{\varphi}' = \vec{\varphi}'_\sigma - \vec{\varphi}' - \langle \vec{\varphi}'_\sigma - \vec{\varphi}', e_* \rangle \vec{\varphi}'$, whence

$$\|\mathcal{Q}_0[\vec{\varphi}'_\sigma]\|_{H^1 \times H^1} \le C|\sigma(s)|\|\vec{\varphi}\|_{H^3}.$$

Finally, recalling $\langle \vec{\varphi}', e_* \rangle = 1$, we have $\langle \vec{\varphi}'_\sigma, e_* \rangle = 1 + \langle \vec{\varphi}'_\sigma - \vec{\varphi}', e_* \rangle$, and

$$|\langle \vec{\varphi}'_\sigma - \vec{\varphi}', e_* \rangle| \le C|\sigma(s)|\|e_*\|_{L^2}\|\vec{\varphi}\|_{H^2}.$$

Note that while we are assured that $|\sigma(s)|$ is appropriately small, $\langle \vec{\varphi}'_\sigma, e_* \rangle = 1 + O(|\sigma(s)|)$ and its reciprocal is well-defined and in fact $\frac{1}{\langle \vec{\varphi}'_\sigma, e_* \rangle} = 1 + O(\sigma(s))$.

In view of this preparatory estimates, we are now ready to setup the non-linear persistence argument as follows. Fix $0 < \beta < \alpha$. Then, fix $\sigma \in (\beta, \alpha)$, say $\sigma = \frac{\alpha+\beta}{2}$. We have then $0 < \beta < \sigma < \alpha$. Applying the estimate (3.17), we obtain for the free solution of (4.5) the bound

$$\|e^{t\mathcal{H}} \vec{v}_0\|_{H^1 \times H^1} \le C_0 e^{-\beta t}\|\vec{v}_0\|_{H^1}.$$

We will show that there exists $\epsilon : 0 < \epsilon << 1$, so that whenever the initial data $\vec{v}_0$ satisfies $C_0\|\vec{v}_0\|_{H^1} \le \frac{\epsilon}{2}$, then the system (4.5) and (4.6) has a global solution, which in addition satisfies the bounds

$$\|v(t, \cdot)\|_{H^1 \times H^1} \le \epsilon e^{-\beta t} \tag{4.7}$$

$$|\sigma'(t)| \le \epsilon^{\frac{3}{2}} e^{-\beta t}. \tag{4.8}$$

It remains to determine whether such an ϵ exists. For any initial data, the system (4.5) and (4.6) certainly has a solution in some time interval, which may be very small, dependent on the data. Denote

$$t_* = \sup\{\tau : \sup_{0 \le t \le \tau} e^{\beta t}\|v(t, \cdot)\|_{H^1 \times H^1} \le \epsilon, \sup_{0 \le t \le \tau} e^{\beta t}\|\sigma'(t)\| \le \epsilon^2\}.$$

Clearly, the problem is to show that for all small enough ϵ, $\tau_* = \infty$. Also, for all $t \in (0, t_*)$, we have $|\sigma(t)| \le \int_0^t |\sigma'(s)|ds \le \frac{\epsilon^{\frac{3}{2}}}{\beta}$.

We are now ready for the bootstrapping step. Taking absolute values in (4.6), we obtain

$$|\sigma'(t)| \le \frac{C}{1 - C|\sigma(t)|}(\|\vec{v}(t)\|_Y^2 + \|\vec{v}(t)\|_Y^3 + |\sigma(t)|\|\vec{v}(t)\|_Y$$

where $Y = H^1 \times H^1$. Using the *a priori* bounds on $\|\vec{v}(s)\|_Y$ and $|\sigma'(s)|$ guaranteed to us by $s \in (0, t_*)$, we have

$$|\sigma'(t)| \le \frac{C}{1 - C\epsilon^{\frac{3}{2}}}(\epsilon^2 e^{-2\beta t} + \epsilon^{\frac{5}{2}} e^{-\beta t}) \le \epsilon^{\frac{3}{2}} e^{-\beta t},$$

for small enough ϵ.

Taking Y norms in (4.5) and applying (3.17) with $\alpha - \delta = \sigma$, yields

$$\begin{aligned} \|\vec{v}(t)\|_Y &\le \frac{\epsilon}{2} e^{-\beta t} + C\int_0^t e^{-\sigma(t-s)}(\|\vec{v}(s)\|_Y^2 + \|\vec{v}(s)\|_Y^3 + |\sigma(s)|\|\vec{v}(s)\|_Y + |\sigma'(s)||\sigma(s)|ds \le \\ &\le \frac{\epsilon}{2} e^{-\beta t} + C\epsilon^{\frac{3}{2}} \int_0^t e^{-\sigma(t-s)}\|\vec{v}(s)\|_Y ds + C\int_0^t e^{-\sigma(t-s)}(\epsilon^2 e^{-2\beta s} + \epsilon^3 e^{-\beta s}) ds. \end{aligned}$$

As a consequence, we obtain the following inequality for $I(t) := e^{\sigma t}\|\vec{v}(t)\|_Y$,

$$I(t) \le \frac{\epsilon}{2} e^{(\sigma-\beta)t} + C\epsilon^{3/2} \int_0^t I(s)ds + C\epsilon^2 e^{(\sigma-\beta)t} \tag{4.9}$$

Denoting $M(t) := \int_0^t I(s)ds$, we have

$$(M(t)e^{-C\epsilon^{3/2}t})' \le C\epsilon e^{(\sigma-\beta-C\epsilon^{3/2})t},$$

whence, assuming $\epsilon : \sigma - \beta - C\epsilon^{3/2} > \frac{\sigma-\beta}{2}$ and taking into account $M(0) = 0$, we have

$$M(t) \le C\epsilon e^{(\sigma-\beta)t}.$$

Plugging this back into (4.9) and after some elementary manipulations, we have the bound

$$\|\vec{v}(t)\|_Y \le \frac{\epsilon}{2} e^{-\beta t} + C\epsilon^{3/2} e^{-\beta t} \le \epsilon e^{-\beta t},$$

for small enough ϵ. Thus, the nonlinear stability is proved in full.

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