EXISTENCE AND STABILITY OF SOLITARY WAVES FOR THE INHOMOGENEOUS NLS

ABBA RAMADAN AND ATANAS G. STEFANOV

ABSTRACT. In this paper, we identify necessary and sufficient conditions for the existence of appropriately localized waves for the inhomogeneous semi-linear Schrödinger equation driven by the subLaplacian dispersion operators $(-\Delta)^s, 0 < s \leq 1$. We construct these waves and we establish sharp asymptotics, both at the singularity 0 and for large values. We show the non-degeneracy of these waves. Finally, we provide spectral and orbital stability classification, under slightly more restrictive assumptions.

1. INTRODUCTION

The main object of consideration in this article will be the Cauchy problem for the fractional inhomogenous nonlinear Schrödinger equation[1] More precisely, we consider

$$\begin{cases} i u_t + (-\Delta)^s u - |x|^{-b}|u|^{p-1} u = 0, (t,x) \in \mathbf{R} \times \mathbf{R}^n, n \geq 1, \\ u(0,x) = u_0(x) \end{cases} \tag{1.1}$$

where we henceforth restrict ourselves to parameters (b,p,s), satisfying the following natural assumptions $b > 0, p > 1, s \in (0,1)$. Our goal in this article is the construction and the stability of solitary waves for (1.1). More specifically, the solitons are in the form of standing waves, that is special solutions in the form $u(x,t) = e^{-i\omega t}\Phi_\omega(x), \phi > 0$. These satisfy the profile equation[2]

$$(-\Delta)^s \Phi + \omega\Phi - |x|^{-b}\Phi^p = 0, x \in \mathbf{R}^n. \tag{1.2}$$

We now turn to a review of the literature regarding the well-posedness results for (1.1).

1.1. **The model - well-posedness results for the classical case** $s = 1$**.** The classical model, $s = 1, b = 0, p > 1$ has been extensively studied in the literature, in terms of well-posedness of the Cauchy problem, long time behavior etc.. As these results are by now classical and well-known, we will not review them here, but we will rather refer the interested reader to the following sources [4, 27, 28, 44, 5, 7, 6, 3, 8, 9, 36, 2].

Recently the well-posedness of (1.1) appeared in the literature for the Laplacian case, i.e. $s = 1$. in fact, Farah [20] proved a Gagliardo-Nirenberg type estimate and use it to establish sufficient conditions for global existence and blow-up in $H^1(\mathbb{R}^n)$ for $\frac{4-2b}{n} < p < \frac{4-2b}{n-2}$ and $0 < b < \min(2,n)$, which was later extended by Dinh [16]. Moreover, Guzmán [31] showed that (1.1) is globally well-posed for the initial data in $H^s(\mathbb{R}^n)$ with $0 \leq s \leq 1$ using the contraction mapping principle based on the Strichartz estimates. In [30], the authors showed the global well-posedness in $H^1(\mathbb{R}^n)$ of (1.1) with $s = 1$, using the assumption that if the initial data u_0

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[1] see Section 2.1 for precise definitions of the fractional derivative operator

[2] The sense in which (1.2) holds is to be made precise later on, see Section 4 below.

satisfies $\|u_0\|_{L^2} < \|\psi\|_{L^2}$, where ψ is the unique positive radial soliton of 1.2. Moreover they also showed strong instability of the standing waves.

In the paper [48], the authors showed the global existence and blow up of solutions in $\mathbb{R}^2$, under various assumptions on the initial data. In addition, the paper [21] showed that if the initial datum $u_0 \in H^1(\mathbb{R}^3)$ satisfies that the momentum as well as the Hamiltonian of (1.1) with $s = 1, n = 3$ is dominated by same conserved quantities of (1.2) similarly, $\|\nabla u_0\|_{L^2}^{\frac{1+b}{2}} \|u_0\|_{L^2}^{\frac{1-b}{2}} < \|\nabla Q\|_{L^2}^{\frac{1+b}{2}} \|Q\|_{L^2}^{\frac{1-b}{2}}$ where Q satisfies (1.2), then the solution u to the Cauchy problem is global in $H^1(\mathbb{R}^3)$ for $0 < b < 1$, and asymptotically linear both forward and backward in time for u_0 radial and $0 < b < 1/2$. In [18], the authors studied the decay properties of global solutions for the equation($s = 1$) when $1 < p < \frac{4-2b}{n-2}$ for $n \geq 3$ and using this they showed the energy scattering for the equation in the case $1 + \frac{4-2b}{n} < p < 1 + \frac{4-2b}{n-2}$. In [11], the authors have studied the global well-posedness for the inhomogeneous NLS, whose scaling critical index $s_c < 0$. Chen, [12] has considered the model (1.1), with non-linearity $|x|^b |u|^{p-1} u, b > 0$. He has identified simple but sharp conditions the solutions exist globally and others, under which the solutions blow up in finite time.

We now turn our attention to the issue of the existence of the solitary waves and their stability.

1.2. **Solitary waves and stability in the classical case** $s = 1$**.** The question for existence of solitary waves (1.2) and their stability was investigated in some specific instances of nonlinearity $g(x, |u|^2)u$ in the late 90's in [35]. Specifying to the case
$V(x)|u|^{p-1}u$, and in particular to the case, $V = V(\epsilon|x|), 0 < \epsilon << 1$ was considered in [22], [43]. A general problem modeled by (1.1), was studied systematically for first time in the work of De Bouard-Fukuizumi, [19]. More precisely, they consider classical NLS (i.e. $s = 1$) with focussing nonlinearity $V(x)|u|^{p-1}u$, where $V \geq 0$,

$$V \in L_{loc.}^{\overline{\frac{2n}{n+2-(n-2)p}}}(\mathbf{R}^n), \quad \lim_{x \to \infty} V(x)|x|^b = 1, \tag{1.3}$$

which of course contains the important case $V(x) = |x|^{-b}$, under the constraints $0 < b < 2, n \geq 3, 1 < p < 1 + \frac{4-2b}{n-2}$. In this work, they show the existence of non-negative solitary wave solutions under the same assumptions. Furthermore, they showed that there exists $\omega_* > 0$, so that the stability of the said solitary waves holds in the range $0 < b < 2, n \geq 3, 1 < p < 1 + \frac{4-2b}{n}$, when the spectral parameter $\omega \in (0, \omega_*)$. The key step in the stability proof is to show that the linear operator associated with the second variation of a Lyapunov functional[3], which is non-degenerate, for this they adapt a method of [41]. The work in a way supplements the earlier work [26], where the instability of the waves was shown in the range $p > 1 + \frac{4-2b}{n}, n \geq 3$, for small enough $\omega > 0$. Further, more general instability results have appeared in [42].

The authors in [32],[29] proved similar results (both for the stable and unstable waves, with frequency ω close to zero), but in the case of non-degeneracy of the linearized operator they employ the spherical of harmonics of the Laplacian.

We now review the fractional case $s < 1$.

1.3. **The fractional case** $0 < s < 1$**.** The case of the fractional Schrödinger operator, that is $s \in (0, 1)$, has also received considerable attention in recent years. Regarding the well-posedness for the standard power non-linearity, we mention the work of Dinh, [17] and the references therein. The paper [46] studied the well-posedness of (1.1) with $b < 0$. Unfortunately, we are not aware of

[3] Although a key assumption, namely $b < 2$ has to be revised to $b < \frac{3}{2}$ in the case $n = 3$, more on this below

any local and global well-posedness results for (1.1). It looks however that the work [10] seems to contain all necessary ingredients in terms of estimates and one has to proceeds as in [20]. We leave this line of investigation open to other researchers.

Regarding solitary waves for the fractional NLS, the real breakthrough came in the article [23], which deals with the case $b = 0, n = 1, s < 1$ about the existence of positive solution of (1.2) has been studied by the authors in [23]. Moreover, the non-degeneracy of the ground state is shown, which plays a very important role in orbital stability of such solutions. In a later work, [24] generalizes the above results in any dimension. More precisely, the uniqueness and non-degeneracy of the ground state solution for $(-\Delta)^s Q + Q - |Q|^{p-1} Q = 0$, with $Q \in H^s(\mathbb{R}^n)$ was established in $\mathbb{R}^n, n \geq 1, s \in (0,1)$ where $1 < p < 1 + \frac{4s}{n-2s}$ for $0 < 2s < n$ and $1 < p < \infty$, $2s \geq n$.

Our goal is to investigate the existence of the waves Φ, given by (1.2), as well as their stability properties. Let us introduce the formally conserved quantities of 1.1:

- the L^2 norm
$$\mathscr{P}[u] = \int_{\mathbb{R}^n} |u(x)|^2 dx$$
- the Hamiltonian
$$\mathscr{H}[u] = \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^n} |x|^{-b} |u(x)|^{p+1} dx$$

We will also make use of the total energy functional, defined as follows
$$E[u] := \mathscr{H}[u] + \frac{\omega}{2} \mathscr{P}[u].$$

In fact, a variant of the local well-posedness theory, presented in Theorem 4.6.6 in [6] for the case $s = 1$, guarantees that for a data $u_0 \in H^s(\mathbf{R}^n)$, $1 < p < 1 + \frac{4s-2b}{n-2s}$, there exists time $T_0 = T_0(\|u_0\|_{H^s})$, so that a strong solution $u(t,\cdot) \in H^s(\mathbf{R}^n)$ to (1.1) exists in $0 < t < T_0$ and moreover $\mathscr{P}(u(t)) = \mathscr{P}(u_0), \mathscr{H}(u(t)) = \mathscr{H}(u_0)$.

Next, we discuss the linearization of (1.1) around the standing waves $e^{-i\omega t} \Phi_\omega$. We perform a standard linearization procedure, namely we take $u = e^{-i\omega t}[\Phi_\omega + v]$, plug it in (1.1) and ignoring the higher order terms $O(v^2)$, we arrive at the linearized system, which after $v = (\Re v, \Im v) =: (v_1, v_2)$ can be written as

$$\begin{pmatrix} \Re v \\ \Im v \end{pmatrix}_t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathscr{L}_+ & 0 \\ 0 & \mathscr{L}_- \end{pmatrix} \begin{pmatrix} \Re v \\ \Im v \end{pmatrix}, \tag{1.4}$$

where the following fractional Schrödinger operators are introduced
$$\begin{aligned} \mathscr{L}_+ &= (-\Delta)^s + \omega - p|x|^{-b} \Phi^{p-1}, \\ \mathscr{L}_- &= (-\Delta)^s + \omega - |x|^{-b} \Phi^{p-1}. \end{aligned}$$

Note that at this point, the properties of the potential $|x|^{-b}\Phi^{p-1}$ are not yet understood, but one has to definitely address the issue of its singularity at zero. This shall be a major concern going forward. We just mention that for the purposes of the stability considerations, it is convenient on using the standard domain $D(\mathscr{L}_\pm) = H^{2s}(\mathbf{R}^n)$, which will lead to some mild additional, perhaps unnecessary, restrictions on the parameters.

Upon the introduction of the operators
$$\mathscr{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathscr{L} := \begin{pmatrix} \mathscr{L}_+ & 0 \\ 0 & \mathscr{L}_- \end{pmatrix},$$

and the assignment $\begin{pmatrix} \Re v \\ \Im v \end{pmatrix} \to e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =: e^{\lambda t}\vec{v}$, we obtain the following time-independent linearized eigenvalue problem

$$\mathscr{J}\mathscr{L}\vec{v} = \lambda\vec{v}. \tag{1.5}$$

1.4. **Main results.** Before we formally state our results, we need a few rigorous definitions about the objects that we study. We employ the following standard definition of stability.

Definition 1. *We say that the wave $e^{-\omega t}\Phi$ is spectrally stable, if the eigenvalue problem* (1.5) *has no non-trivial solutions $(\lambda, \vec{v})$, with $\Re\lambda > 0$. Otherwise, in the cases where there is $\lambda : \Re\lambda > 0$ and $\vec{v} \neq 0$, so that* (1.5) *is satisfied, we say that the wave $e^{-i\omega t}\Phi$ is spectrally unstable and λ is referred to as an unstable mode for* (1.5)

We say that $e^{-i\omega t}\Phi$ is orbitally stable, if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$, so that whenever $\|u_0 - \Phi\|_{H^s(\mathbf{R}^n)} < \delta$, then

- *The solution u of* (1.1)*, in appropriate sense, with initial data $u_0 \in H^s$ is globally in $H^s(\mathbf{R}^n)$, i.e. $u(t,\cdot) \in H^s(\mathbf{R}^n)$.*
- $$\sup_{t>0}\inf_{\theta\in\mathbf{R}} \|u(t,\cdot) - e^{i(\omega t+\theta)}\Phi(\cdot)\|_{H^s(\mathbf{R}^n)} < \epsilon.$$

Key Assumptions

Let Φ be a solution of (1.2). We assume that:

(1) The solution map $g \to u_g$ has continuous dependence on initial data property in a neighborhood of Φ. That is, there exists $T_0 > 0$, so that for all $\epsilon > 0$, there exists $\delta > 0$, so that whenever $g : \|g - \Phi\|_{H^s} < \delta$, then $\sup_{0<t<T_0} \|u_g(t,\cdot) - e^{-i\omega t}\Phi_\omega\|_{H^s} < \epsilon$.
(2) All initial data, sufficiently close to Φ_ω in H^s norm, generates a global in time solution u_g of (1.1). In addition, the L^2 norm and the Hamiltonian for these solutions are conserved. That is
$$\mathscr{P}[u_g(t)] = \mathscr{P}[g], \mathscr{H}[u_g(t)] = \mathscr{H}[g].$$

Remarks:

- The continuity dependence on initial data property stated above is a simple consequence of a standard local well-posedness result, in the spirit of Theorem 4.6.4, [6]. Since such result seems unavailable at the moment, we explicitly assume its veracity.
- There is also the notion of asymptotic stability for our waves. We do not formally introduce herein, as we do not have definite results in this direction. We conjecture it to be true, in all cases of spectral/orbital stability listed in our main theorems below.

Next, we introduce a subset in the parameters space (n, p, s, b), which will be helpful in the sequel

$$\mathscr{A} := \begin{cases} n = 1,\ \frac{1}{2} \le s < 1,\ 0 < b < 1,\ 1 < p < \infty \\ n = 1,\ s \in (0, \frac{1}{2}),\ 0 < b < 2s,\ 1 < p < 1 + \frac{4s-2b}{1-2s} \\ n \ge 2,\ s \in (0,1),\ 0 < b < 2s,\ 1 < p < 1 + \frac{4s-2b}{n-2s} \end{cases}.$$

This set will turn out to describe the necessary and sufficient conditions under which Φ_ω exists.

Our first theorem is a sufficiency result for the existence of the solitary waves Φ_ω.

Theorem 1. *(Existence results)*

Let $(n,p,s,b)\in\mathscr{A}$, $\omega>0$. *Then, there exits a bell-shaped function*[4] $\Phi_\omega\in H^s(\mathbf{R}^n)\cap L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$, *so that the equation* (1.2) *is satisfied in a distributional sense and also in the strong sense*

$$\Phi_\omega=((-\Delta)^s+\omega)^{-1}[|x|^{-b}\Phi_\omega^p]. \tag{1.6}$$

Finally, under the assumption $s\in(\frac{1}{2},1]$, *we have that* $\phi\in C^1(\mathbf{R}^n\setminus\{0\})$.

Remark: We have in fact much more precise description about the behavior of $\phi,\nabla\phi$ in Proposition 4 below.

Interestingly, we have the appropriate converse statement, which makes $\mathscr{A}$ the necessary and sufficient set of requirements for the solvability of (1.2).

Theorem 2. *Assume that a positive function* $\psi\in H^s(\mathbf{R}^n)\cap L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$ *satisfies*

$$(-\Delta)^s\psi+\omega\psi=|x|^{-b}\psi^p$$

in a distributional sense. Then $(n,p,s,b)\in\mathscr{A}$ *and* $\omega>0$.

Next, we are concerned with the stability of the waves constructed in Theorem 1.

Theorem 3. *Let* $(n,p,s,b)\in\mathscr{A}$ *and* $\omega>0$. *In addition, assume that* $2b<n$ *and* $s\in(\frac{1}{2},1]$. *Let* Φ_ω *be the solution constructed in Theorem 1. Then,*

(1) the linearized operators $\mathscr{L}_\pm, D(\mathscr{L}_\pm)=H^{2s}(\mathbf{R}^n)$ *are self-adjoint and* $\Phi_\omega\in D(\mathscr{L}_+)$.
(2) Φ_ω *non-degenerate, in the sense that* $Ker[\mathscr{L}_+]=\{0\}$.

For $1<p<1+\frac{4s-2b}{n}$ *the soliton* $e^{-i\omega t}\Phi_\omega$ *is spectrally and orbitally stable. In the complementary range,*

$$1+\frac{4s-2b}{n}<p<\begin{cases}\infty & n=1\\ 1+\frac{4s-2b}{n-2s} & n\geq 2,\end{cases}$$

the soliton is spectrally unstable.

Remarks:

(1) According to the necessity statements in Theorem 2, the results in Theorem 3 provide a full classification of the bell-shaped solutions that exists, in the cases $s\in(\frac{1}{2},1)$ and $2b<n$. Note that the constraint $2b<n$ is already contained in the necessity assumption for $n\geq 4$.
(2) In the case $n=3$, the constraint $b<\frac{3}{2}$ is slightly worse than the necessity assumptions, $b<2$. This was the claim in [19], but one certainly faces some difficulties (specifically with $D(\mathscr{L}_+)$) in the range $b\in(\frac{3}{2},2)$. See the remarks after Corollary 2 below.
(3) Our results seem to be new even in the case $s=1$, in low dimensions, $n=1,2$. The restrictions $b<\frac{1}{2}$ for $n=1$ and $b<1$ for $n=2$ are more restrictive than the necessary assumptions $(n,p,s,b)\in\mathscr{A}$. It is interesting whether one can establish rigorously the stability situation for these parameters. As we discuss at length, the main issue is to make sense of the functional analytic framework, in particular the domains of the linearized operators $\mathscr{L}_\pm$.
(4) The case $p=\frac{4s-2b}{n}$ is a bifurcation case, where one gets a crossing through zero of a pair of purely imaginary eigenvalues to a pair of stable/unstable real eigenvalues. This is also where the equation (1.1) enjoys an extra, so called pseudo-conformal symmetry, hence the extra pair of eigenvalues at zero. Even though one has spectral stability for this case, one generally expects the corresponding waves to be spectrally unstable, as in the classical NLS, see the seminal paper [14] for details.

[4] That is, a radial function, which is non-increasing in the radial variable

The paper is planed as follows. In section 2, we give some necessary preliminaries such as function spaces, asymptotics of the Green's functions for the fractional Laplacian, the basics of rearrangements and a weighted Sobolev inequality. In Section 3, we introduce the Pohozaev's identities, which in turn imply the necessary conditions for the existence of the waves, which is the content of Theorem 2. In Section 4, we present the variational construction of the waves along with some further properties of the profiles, such as boundedness, sharp asymptotics at zero and smoothness. In Section 5, we provide a self-adjoint realization of the linearized operators $\mathscr{L}_\pm$, followed by some preliminary coercivity properties. We also introduce the Frank-Lenzman-Silvestre Sturm oscillation theory for fractional Schrödinger operators as well as an adaptation of their method to our situation, which has to address singular potentials in the next section. In Section 6, we establish the non-degeneracy of the waves. This requires decomposition in spherical harmonics and careful analysis on the radial subspace by using the Frank-Lenzman-Silvestre theory developed in the previous section as well as an argument to rule out non-trivial elements in the first harmonic subspace. In Section 7, we provide a short introduction to the index counting theory, which provide an useful criteria for spectral stability. In Propositions 10 and 11, we show the coercivity of $\mathscr{L}_\pm$ on $\{\Phi\}^\perp$, which is an important ingredient of the orbital stability scheme. Finally, we show the orbital stability (whenever spectral stability holds) in Proposition 12.

2. Preliminaries

2.1. **Function spaces, Fourier transform and basic operators.**

In order to fix the notations, we shall use the standard expressions for $\|\cdot\|_{L^p(\mathbf{R}^n)}, 1 \le p \le \infty$ as well as the following expression for the Fourier transform and its inverse

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx, \quad f(x) = \int_{\mathbf{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi.$$

The operators $(-\Delta)^s, 0 < s < 1$ are defined in a classical way on the Schwartz class[5] $\mathscr{S}$ via $\widehat{(-\Delta)^s f}(\xi) = (2\pi|\xi|)^{2s}\hat{f}(\xi)$. Accordingly, the Sobolev spaces are taken $\|f\|_{\dot{H}^s} := \|(-\Delta)^{s/2} f\|_{L^2}$, $\|f\|_{H^s} = \|f\|_{\dot{H}^s} + \|f\|_{L^2}$. More generally, the Sobolev spaces $W^{\alpha,p}, \alpha > 0, 1 < p < \infty$ are introduced as completions of the Schwartz family in the norms $\|f\|_{W^{\alpha,p}} := \|(-\Delta)^{s/2} f\|_{L^p} + \|f\|_{L^p}$. The use of weighted spaces is necessitated by the context, so we introduce

$$\|f\|_{\dot{L}^{q,-b}} = \left(\int_{\mathbf{R}^n} |x|^{-b} |f(x)|^q dx\right)^{1/q}.$$

The following commutator identity, see p. 1703, [24], will be of special interest

$$[(-\Delta)^s, x\cdot\nabla_x] = 2s(-\Delta)^s. \tag{2.1}$$

We will also need properties of the kernel of the operator $(I + (-\Delta)^s)^{-1}, s > 0.$. We state a precise result next.

Lemma 1. *Let $0 < s < 1$. Then, the function $G_s(x) : \widehat{G_s}(\xi) = (1 + (4\pi^2|\xi|^2)^s)^{-1}$ satisfies*

- *There is $C = C_{s,n}$, so that*

$$G_s(x) \le C_{s,n}|x|^{-n}$$

when $|x| > 1$,

[5] and then by extension in any Banach space for which $\mathscr{S}$ is a dense subspace

- *For* $|x| \le 1$*, there is*

$$G_s(x) \sim \begin{cases} |x|^{2s-n} + O(1) & 2s < n \\ \ln(2/|x|) + o(x) & 2s = n \\ 1 + o(x) & 2s > n \end{cases}$$

- $G_s > 0$, $G_s \in L^1(\mathbf{R}^n)$.

Regarding ∇G_s*, we have the following bounds, in the regime* $2s < n$

$$|\nabla G_s(x)| \le C \begin{cases} |x|^{-n-1} & |x| > 1 \\ |x|^{2s-n-1} & |x| \le 1 \end{cases} \tag{2.2}$$

Proof. First, take a partition of unity, so that there is a function φ, supported in $\{\xi : |\xi| < 1\}$ and $\zeta(\xi) := \varphi(\xi) - \varphi(2\xi)$, whence $\varphi(\xi) + \sum_{k=1}^{\infty} \zeta(2^{-k}\xi) = 1$. Let $|x| > 1$, say $|x| \sim 2^l, l \ge 0$. We have the partition of unity

$$1 = \varphi(2^l \xi) + (1 - \varphi(2^l \xi)) = \varphi(2^l \xi) + \sum_{k=1-l}^{\infty} \zeta(2^{-k}\xi)$$

whence

$$\begin{aligned} G_s(x) &= \int \frac{1}{1 + (4\pi^2|\xi|^2)^s} e^{-2\pi i x\cdot\xi} d\xi = \int \frac{1}{1 + (4\pi^2|\xi|^2)^s} e^{-2\pi i x\cdot\xi} \varphi(2^l\xi) d\xi + \\ &+ \sum_{k=1-l}^{\infty} \int \frac{1}{1 + (4\pi^2|\xi|^2)^s} e^{-2\pi i x\cdot\xi} \zeta(2^{-k}\xi) d\xi. \end{aligned}$$

In the first integral, we estimate the integrand by absolute value, whence we obtain the bound $C2^{-ln} \sim |x|^{-n}$. For a given x, we identify $j \in [1, n]$, so that $|x_j| \ge \frac{2^l}{n}$. Integrating by parts N times in the variable x_j (and $N > n+1$) and taking absolute values implies a bound

$$\sum_{k=1-l}^{\infty} \frac{1}{(2^k |x_j|)^N} 2^{kn} \lesssim 2^{-ln} \sim |x|^{-n}.$$

For $|x| < 1$, let us consider the case $2s < n$, as the others are similar and somewhat simpler. Say $|x| \sim 2^{-l}, l \ge 0$. We now use the partition of unity

$$1 = \varphi(2^{-l}\xi) + \sum_{k=l+1}^{\infty} \zeta(2^{-k}\xi)$$

Again, for the integral with $\varphi(2^{-l}\xi)$ we estimate by the absolute values

$$\left| \int \frac{1}{1 + (4\pi^2|\xi|^2)^s} e^{-2\pi i x\cdot\xi} \varphi(2^{-l}\xi) d\xi \right| \le C 2^{l(n-2s)} \sim |x|^{2s-n},$$

while for the other integrals, we again integrate by parts N times in $|x_j| \ge \frac{2^{-l}}{n}$. The estimates are again

$$\sum_{k=l+1}^{\infty} \frac{1}{(2^k |x_j|)^N} 2^{k(n-2s)} \le C 2^{l(n-2s)} \sim |x|^{2s-n}.$$

For ∇G_s, the bounds (2.2) follow in an identical manner, once we recognize that taking derivatives results in an extra power of $|x|^{-1}$.

The statement $G_s > 0$ (and in fact G_s is bell-shaped), can be proved via the representation

$$\frac{1}{1 + (4\pi^2|\xi|^2)^s} = \int_0^{\infty} e^{-t(1+(4\pi^2|\xi|^2)^s)} dt = \int_0^{\infty} e^{-t} e^{-t(4\pi^2|\xi|^2)^s} dt$$

and the well-known fact that $\widehat{e^{-|\xi|^{2s}}}$ is a bell-shaped function, as long as $0 < s \leq 1$. Thus,

$$\|G_s\|_{L^1} = \int G_s(x)dx = \hat{G}_s(0) = 1.$$

$\square$

2.2. **Rearrangements.** In this subsection, we discuss the techniques of rearrangements.

Let A be a measurable set of finite volume in $\mathbb{R}^n$. Its symmetric rearrangement A^* is the open centered ball whose volume agrees with A i.e $A^* = \{x \in \mathbb{R}^n : |\omega_n||x|^n < Vol(A)\}$. We say that a measurable $f \geq 0$ vanish at infinity if for all t we have $\mu(\{x : f > t\}) < \infty$.

Definition 2. *let $f \geq 0$ be a measurable function that vanish at infinity we define define the symmetric decreasing rearrangement f^* of f by symmetrizing its the level set $f^*(x) = \int_0^\infty \chi_{\{f(x)>t\}^*} dt$ and is uniquely determined by $f^*(t) = \inf\{s > 0 : d_f(s) \leq t\}$*

And its bell-shaped rearrangement $f^\#$ on $\mathbb{R}$, as $f^\#(t) = f^*(2|t|)$.

Next, we state the Polya-Szegö inequalities, which will be instrumental in our approach.

Lemma 2. *For $\beta \in (0,1)$ and $f \in H^\beta(\mathbf{R}^n)$, its decreasing rearrangement $f^* \in H^\beta(\mathbf{R}^n)$ and*

$$\|(-\Delta)^{\frac{\beta}{2}} f\|_{L^2} \geq \|(-\Delta)^{\frac{\beta}{2}} f^*\|_{L^2} \tag{2.3}$$

Our next proposition deals with a control of the weighted norms appearing in (3.2) in terms of a Sobolev embedding.

2.3. **Weighted Sobolev inequality.**

Proposition 1. *For either one of the cases,*

- $n = 1, \sigma \in [\frac{1}{2}, 1)$, $0 < a < 1$, $2 \leq q < \infty$,
- $n = 1, 0 < \sigma < \frac{1}{2}$, $0 < a < 2\sigma$, $2 \leq q < 2 + \frac{4\sigma - 2a}{1-2\sigma}$,
- $n \geq 2$, $0 < \sigma < 1$, $0 < a < 2\sigma$, $2 \leq q < 2 + \frac{4\sigma-2a}{n-2\sigma}$,

there exists C, depending on all parameters, so that

$$\left(\int_{\mathbf{R}^n} |x|^{-a}|\phi|^q dx\right)^{\frac{1}{q}} \leq C\|\phi\|_{H^\sigma(\mathbf{R}^n)}. \tag{2.4}$$

Remark: Note that the assumptions in Proposition 1 ensure that $a < n$. Also, for $q = 2$, there is the estimate

$$\left(\int_{\mathbf{R}^n} |x|^{-a}|\phi|^2 dx\right)^{\frac{1}{q}} \leq C_\epsilon \|\phi\|_{H^{\frac{a}{2}+\epsilon}(\mathbf{R}^n)}, \tag{2.5}$$

for every $\epsilon > 0$.

Proof. For the case $n \geq 2$, $\sigma > 0$, $0 < a < 2\sigma$, and $2 \leq q < 2 + \frac{4\sigma-2a}{n-2\sigma}$, we proceed as follows. By Sobolev embedding, we have (since $n\left(\frac{1}{2} - \frac{1}{q}\right) < \sigma$)

$$\left(\int_{|x|>1} |x|^{-a}|\phi|^q dx\right)^{\frac{1}{q}} \leq \left(\int_{|x|>1} |\phi|^q dx\right)^{\frac{1}{q}} \leq C\|\phi\|_{L^q} \leq C\|\phi\|_{H^\sigma}.$$

Next, for $|x| < 1$

$$\left(\int_{|x|<1} |x|^{-a}|\phi|^q dx\right)^{\frac{1}{q}} \leq C\left(\sum_{j=0}^\infty 2^{ja} \int_{|x|\sim 2^{-j}} |\phi|^q dx\right)^{\frac{1}{q}}$$

And by Holder inequality we have for every $r \geq q$,

$$\int_{|x|\sim 2^{-j}} |\phi|^q \leq \left(\int |\phi|^r\right)^{\frac{q}{r}} (2^{-jn})^{(1-\frac{q}{r})}.$$

Thus

$$\left(\int_{|x|<1} |x|^{-a}|\phi|^q dx\right)^{\frac{1}{q}} \leq \left(\sum_{j=0}^{\infty} (2^{-jn})^{(1-\frac{q}{r})+ja} \|\phi\|^q_{L^r(|x|\sim 2^{-j})}\right)^{\frac{1}{q}}.$$

Select *any* $r \in (q, \infty)$, so that

$$a < n\left(1 - \frac{q}{r}\right),\ \ n\left(\frac{1}{2} - \frac{1}{r}\right) < \sigma$$

That is,

$$\frac{1}{2} - \frac{\sigma}{n} < \frac{1}{r} < \frac{1-\frac{a}{n}}{q},$$

which is possible, due to the restriction $2 \leq q < 2 + \frac{4\sigma-2a}{n-2\sigma}$. We have

$$\begin{aligned} &\left(\sum_{j=0}^{\infty} (2^{-jn})^{(1-\frac{q}{r})+ja} \|\phi\|^q_{L^r(|x|\sim 2^{-j})}\right)^{\frac{1}{q}} = \left(\sum_{j=0}^{\infty} (2^{j(a-n(1-\frac{q}{r}))} \|\phi\|^q_{L^r(|x|\sim 2^{-j})}\right)^{\frac{1}{q}} \leq \\ \leq\ \ & C_r \sup_j \|\phi\|_{L^r(|x|\sim 2^{-j})} \leq C_r \|\phi\|_{L^r} \leq C_r \|\phi\|_{H^{n(\frac{1}{2}-\frac{1}{r})}} \leq C_r \|\phi\|_{H^\sigma}. \end{aligned}$$

where in the last step we have used the Sobolev embedding and $n\left(\frac{1}{2} - \frac{1}{r}\right) < \sigma$. The case $n = 1, \sigma \in (0, \frac{1}{2}), a < 2\sigma, 2 \leq q < 2 + \frac{4\sigma-2a}{1-2\sigma}$ is done in an identical manner.

For the case $n = 1, \sigma \geq \frac{1}{2}, 2 \leq q < \infty$ is as follows. By Sobolev embedding $H^\sigma(\mathbf{R}) \hookrightarrow L^q(\mathbf{R})$, so

$$\left(\int_{|x|>1} |x|^{-b}|\phi|^q dx\right)^{\frac{1}{q}} \leq \left(\int_{|x|>1} |\phi|^q dx\right)^{\frac{1}{q}} \leq C\|\phi\|_{H^\sigma}.$$

The term $\left(\int_{|x|<1} |x|^{-b}|\phi|^q dx\right)^{\frac{1}{q}}$ is controlled in the same way as above, we omit the details. $\square$

Remark: An easy formulation of the requirements in Corollary 1 would be to say that the parameters $(n, q-1, \sigma, a)$ belong to the set $\mathscr{A}$.

3. Necessary conditions for the waves: proof of Theorem 1

The approach for the proof of Theorem 1 is to exploit the scaling and the associated Pohozaev's identities, which in due course will lead us to the set of constraints $\mathscr{A}$.

3.1. Pohozaev identities and consequences.

Before we make assumptions on the smoothness and decay properties of the profiles ϕ, and in addition the sense in which (1.2) is satisfied, (1.2) remains a formal object. In order to further demystify the ranges in which one might expect reasonable solutions of (1.2), we provide the following Pohozaev type identities.

Lemma 3. *(Pohozaev identities) Assume that* $0 < b < n$ *and* $\psi \in H^s(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$, *with* $\psi > 0$ *satisfies*

$$(-\Delta)^s \psi + \omega\psi - |x|^{-b}\psi^p = 0 \tag{3.1}$$

in a distributional sense. Then,

$$\int_{\mathbf{R}^n} |x|^{-b}\psi^{p+1}dx = \frac{2ws(p+1)}{2(n-b)-(n-2s)(p+1)}\int_{\mathbf{R}^n}\psi^2 dx, \tag{3.2}$$

$$\int_{\mathbf{R}^n} |(-\Delta)^{s/2}\psi|^2 dx = \frac{w(n(p+1)-2(n-b))}{2(n-b)-(n-2s)(p+1)}\int_{\mathbf{R}^n}\psi^2 dx. \tag{3.3}$$

$$\omega\int_{\mathbf{R}^n}\psi(x)dx = \int_{\mathbf{R}^n}|x|^{-b}\psi^p dx. \tag{3.4}$$

Proof. A formal proof (i.e. one where we assume that ψ has enough smoothness and decay properties) is as follows. Take a dot product with ψ in (3.1) and integrating by part we get

$$\int |(-\Delta)^{s/2}\psi|^2 dx + \omega\int \psi^2(x)dx = \int |x|^{-b}\psi^{p+1}(x)dx$$

If we take a dot product with $x\cdot\nabla_x\psi = \sum_{j=1}^n x_j\partial_j\psi$, taking into account the commutation formula (2.1) and various integration by parts calculations, we obtain another relation between $\int|(-\Delta)^{s/2}\psi|^2dx$ and $\int|x|^{-b}\psi^{p+1}(x)dx$, namely

$$(s-\frac{n}{2})\int |(-\Delta)^{s/2}\psi|^2dx + \frac{n-b}{p+1}\int |x|^{-b}\psi^{p+1}(x)dx = \frac{n\omega}{2}\int\psi^2(x)dx.$$

Solving the last two relations for $\int|(-\Delta)^{s/2}\psi|^2dx, \int|x|^{-b}\psi^{p+1}(x)dx$, we obtain (3.2), (3.3). Integrating (3.1) yields (3.4).

For ψ, which is not necessarily smooth and decaying, one follows similar scheme. To establish (3.2), test the equation (3.1) by a sequence of Schwartz function ψ_N with $\lim_k \|\psi_N-\psi\|_{H^s(\mathbf{R}^n)\cap L^1(\mathbf{R}^n)} = 0$ and then take limits. In order to show (3.3), test (3.1) by $x\cdot\nabla\psi_N$. Again taking into account the commutation relation $[(-\Delta)^s, x\cdot\nabla] = 2s(-\Delta)^s$ and taking limits as $\psi_N\to\psi$ establishes (3.3). The formula (3.4) is proved after testing (3.1) by a function $\chi(x/N), N>>1$ (where χ is compactly supported and $\chi(x)=1, |x|<1$) and taking limits $N\to\infty$. □

Implicit in the formulas (3.2), (3.3) displayed above is that the parameters need to satisfy certain conditions, so that ψ exists. We collect the necessary conditions in the following corollary.

Corollary 1. *Let $p>1$, $n\geq 1, s\in(0,1), b>0$. If ψ with properties listed in Lemma 3 exist, then $\omega>0$ and the parameters must satisfy one of the following relations:*

- $n=1$, $s\in[\frac{1}{2},1), 0<b<1, 1<p<\infty$.
- $n=1, 0<s<\frac{1}{2}, b<2s$,
$$1<p<1+\frac{4s-2b}{1-2s}$$
- $n\geq 2$, $b<2s$,
$$1<p<1+\frac{4s-2b}{n-2s}. \tag{3.5}$$

Remark: Corollary 1 simply states that if ψ solves (3.1), then $(n,p,s,b)\in\mathscr{A}$.

Proof. The fact that $\omega>0$ follows from (3.4). If $\psi(0)>0$ and the integral on the left-hand side of (3.2) exists, it is non-singular at zero and hence $b<n$.

From the positivity of the left-hand sides of (3.2), (3.3) and $n(p+1)-2(n-b) = n(p-1)+2b>0$, it follows that $2(n-b)-(n-2s)(p+1)>0$. In particular, for $n=1$, the conditions are satisfied if $s\geq\frac{1}{2}, 1<p<\infty$ or $0<s<\frac{1}{2}$, but then $2s>b$, $1<p<1+2\frac{2s-b}{1-2s}$.

For $n \geq 2$, note that we always have $n - 2s > 0$, whence we come up with $b < 2s$ and (3.5). □

Clearly, Corollary 1 establishes Theorem 2.

4. The Variational Construction and properties of the minimizers

We start with some elementary observations, which will identify conditions under which an important variational problem is well-posed.

4.1. **Well-posedness of the variational problem.** Consider the following functional

$$I_\omega[u] = \frac{\int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 + \omega \int_{\mathbb{R}^n} u^2}{\left(\int_{\mathbb{R}^n} |x|^{-b} |u|^{p+1}\right)^{\frac{2}{p+1}}}.$$

We shall henceforth assume[6] that $b < n$, $\omega > 0$ and $0 < s < 1$. So, for any function $u \in H^s(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) : u \neq 0$, we have that $0 < \int_{\mathbb{R}^n} |x|^{-b} |u|^{p+1} dx < \infty$, so that the ratio $I_\omega[u]$ is well-defined. Since for $u \in \mathscr{S}$ For every $u \neq 0$, $I_\omega[u] > 0$, we will consider the non-negative scalar function

$$m(\omega) := \inf_{u \in \mathscr{S}} I_\omega[u].$$

In the case when the parameters ensure that $m(\omega) > 0$, will be referred to well-posedness, versus the trivial case $m(\omega) = 0$ (which is certainly possible for certain parameter ranges) will be referred to as lack of well-posedness or ill-posedness. We have the following elementary lemma.

Lemma 4. *Assume that $m(1) > 0$. Then,*

$$m(\omega) = m(1) \omega^{\frac{(n-2s)}{2s(p+1)} [p - (1 + \frac{4s-2b}{n-2s})]}. \tag{4.1}$$

In addition, if ϕ is a minimizer for $I_1[u] \to \min$, i.e. $m(1) = I_1(\phi)$, then $\phi_\omega(x) := \phi(\omega^{\frac{1}{2s}} x)$ is a minimizer for $I_\omega[u] \to \min$.

Proof. Take $\phi(x) = \psi(\lambda x)$ then

$$I_\omega[\phi] = \frac{\lambda^{-n+2s} \|(-\Delta)^{s/2} \psi\|^2 + \omega \lambda^{-n} \|\psi\|^2}{\lambda^{2(\frac{n-b}{p+1})} \left(\int_{\mathbf{R}^n} |x|^{-b} \psi^{p+1}\right)^{\frac{2}{p+1}}}.$$

Taking $\omega = \lambda^{2s}$ implies the formula

$$I_\omega[\phi] = \omega^{\frac{-n+2s-\frac{2(n-b)}{p+1}}{2s}} I_1(\psi),$$

whence the formula (4.1) follows by straightforward algebraic manipulations. □

Remarks:

- As was have discussed above, the well-posedness is equivalent to $m(1) > 0$. So far, we have not addressed this issue in a satisfactory manner. Lemma 4 just establishes that m is a specific power function, if the functional I_ω is bounded from a positive constant.
- Note however that under the standing assumptions $s > 0$, $p > 1$, the power of ω appearing in (4.1) is negative exactly when $(n, p, s, b) \in \mathscr{A}$.

[6]and in fact, we shall pose some more restrictions later on

4.2. **Existence of minimizers.** Our next goal is to obtain an existence result, which holds precisely when $(n,p,s,b)\in\mathscr{A}$. As is clear from Proposition 1, it suffices to consider the case $\omega=1$.

Proposition 2. *Let $(n,p,s,b)\in\mathscr{A}$. Then the unconstrained minimization problem*

$$I_\omega[u]\to\min \tag{4.2}$$

has a bell-shaped solution $\phi\in H^s(\mathbf{R}^n)\cap L^{p+1,-b}$, in particular $m(\omega)>0$.

If ϕ is a minimizer of (4.2), with $\|\phi\|_{L^{p+1,-b}}=1$, then ϕ satisfies the Euler-Lagrange equation

$$(-\Delta)^s\phi+\omega\phi-m(\omega)|x|^{-b}\phi^p=0 \tag{4.3}$$

in the following weak sense: for each $h\in C_0^\infty(\mathbf{R}^n\setminus\{0\})$, there is $\langle(-\Delta)^s\phi+\omega\phi-m(\omega)|x|^{-b}\phi^p,h\rangle=0$. Finally, for the linearized operator,

$$\mathscr{L}_+=(-\Delta)^s+\omega-pm(\omega)|x|^{-b}\phi^{p-1},$$

we have that for each real-valued $h\in C_0^\infty(\mathbf{R}^n\setminus\{0\}):\int|x|^{-b}\phi^p(x)h(x)dx=0$, $\langle\mathscr{L}_+h,h\rangle\geq0$.

Remark:

- Proposition 2 does not claim the boundedness of the minimizer ϕ, i.e. the possibility that $\lim_{x\to0}\phi(x)=\infty$ is left open.
- Related to the previous point, the Euler-Lagrange equation may have a significant singularity at zero, due to the presence of $|x|^{-b}$ and the possible singularity of ϕ at zero. We sidestep the issue for the moment, by testing (4.3) away from zero as $h\in C_0^\infty(\mathbf{R}^n\setminus\{0\})$.
- The non-negativity property of $\mathscr{L}_+$ over the set $h\in C_0^\infty(\mathbf{R}^n\setminus\{0\})$, $h\perp|x|^{-b}\phi^p$, normally would indicate that $\mathscr{L}_+$ has at most one negative eigenvalue. This would eventually turn out to be the case, see Proposition 5. Here, we are forced to restrict to a restricted set of test functions, namely $h\in C_0^\infty(\mathbf{R}^n\setminus\{0\})$, as we have not yet resolved the issue with the singularity of the potential $x\to|x|^{-b}\phi^p(x)$ at zero.

Proof. By the arguments in Lemma 4, it suffices to consider the case $\omega=1$. By the assumption $(n,p,s,b)\in\mathscr{A}$, it follows from Proposition 1

$$\left(\int|x|^{-b}\phi^{p+1}\right)^{\frac{2}{p+1}}\leq C\|\phi\|_{H^s}^2,$$

whence

$$\inf_{u\neq0}I_1[u]\geq C^{-1}.$$

Thus, the variational problem (4.2) is well-posed or equivalently $m(1)>0$.

We now need to show that (4.2) actually has a solution. To that end, observe that by the Polya-Szegö inequality (2.3), $\|(-\Delta)^{s/2}u\|\geq\|(-\Delta)^{s/2}u^*\|$ and since

$$\|\phi^*\|_{L^2}=\|\phi\|_{L^2},\ \int_{\mathbf{R}^n}|x|^{-b}|\phi(x)|^{p+1}dx\leq\int_{\mathbf{R}^n}|x|^{-b}|\phi^*(x)|^{p+1}dx,$$

we conclude that $I_1[u]\geq I_1[u^*]$, which implies that we can reduce the set of possible minimizers to the set of bell-shaped functions, i.e. $\{u\in H^s(\mathbf{R}^n)\cap L^{p+1,b}(\mathbf{R}^n):u=u^*\}$. Next, by the dilation property of the functional $I_1(u)=I_1(au)$, we can without loss of generality further reduce to the set $\int_{\mathbf{R}^n}|x|^{-b}u^{p+1}(x)dx=1$.

So, assume that ϕ_k is a minimizing sequence of bell-shaped functions, subject to the constraint $\int_{\mathbf{R}^n}|x|^{-b}\phi_k^{p+1}(x)dx=1$. It follows that

$$\lim_k\|(-\Delta)^{s/2}\phi_k\|_{L^2}^2+\|\phi_k\|_{L^2}^2=m(1). \tag{4.4}$$

We will show that a subsequence of ϕ_k converges in the strong $H^{s/2}(\mathbf{R}^n)$ sense to a minimizer u, which we will show is the desired solution to the minimization problem (4.2). By weak compactness, we have that a subsequence of ϕ_k (which we will assume without loss of generality is ϕ_k itself) tends weakly in $H^{s/2}(\mathbf{R}^n)$ to a function ϕ, which is also trivially bell-shaped.

Since, for bell-shaped functions u we have the pointwise bound for each $x : |x| = R$,

$$|u(x)|^2 \le |B_n|^{-1} R^{-n} \int_{|y|\le R} |u(y)|^2 dy \le |B_n|^{-1} |x|^{-n} \|u\|_{L^2}^2. \tag{4.5}$$

Based on this, we claim that (a subsequence of) ϕ_k converges to ϕ strongly in the topology of $L^{p+1,-b}$. This will follow from the Kolmogorov-Relich-Riesz criteria for compactness in L^p spaces from $\sup_k \|\phi_k\|_{H^{s/2}(\mathbf{R}^n)} < \infty$ (which is a corollary of (4.4)) and once we establish

$$\lim_N \sup_k \int_{|x|>N} |x|^{-b} |\phi_k(x)|^{p+1} dx = 0 \tag{4.6}$$

Indeed, (4.6) follows from the pointwise bounds for bell-shaped functions (4.5), since

$$\sup_k \int_{|x|>N} |x|^{-b} |\phi_k(x)|^{p+1} dx \le C_n \sup_k \|\phi_k\|_{L^2}^{p+1} \int_{|x|>N} |x|^{-b-(p+1)\frac{n}{2}} dx \le C_n N^{-b-\frac{p-1}{2}n} \sup_k \|\phi_k\|_{L^2}^{p+1},$$

which clearly converges to zero as $N \to \infty$. Thus, up to a subsequence $\|\phi_k - \phi\|_{L^{p+1,-b}} \to 0$, whence $\int_{\mathbf{R}^n} |x|^{-b} \phi^{p+1}(x) dx = 1$. In particular, $I_1(\phi) = \|(-\Delta)^{s/2}\phi\|_{L^2}^2 + \|\phi\|_{L^2}^2 \ge m(1)$.

Now, we have by the lower semicontinuity of the weak convergence in $H^{s/2}$ and (4.4) that

$$m(1) \le \|(-\Delta)^{s/2}\phi\|_{L^2}^2 + \|\phi\|_{L^2}^2 \le \liminf_k \|(-\Delta)^{s/2}\phi_k\|_{L^2}^2 + \|\phi_k\|_{L^2}^2 = m(1).$$

It follows that $\lim_k \|(-\Delta)^{s/2}\phi_k\|_{L^2}^2 + \|\phi_k\|_{L^2}^2 = \|(-\Delta)^{s/2}\phi\|_{L^2}^2 + \|\phi\|_{L^2}^2$, whence by the uniform convexity of $\|\cdot\|_{L^2}$

$$\lim_k \|\phi_k - \phi\|_{H^{s/2}(\mathbf{R}^n)} = 0.$$

We conclude that $I_1[\phi] = m(1)$ and ϕ is a solution to (4.2).

Next, we discuss the Euler-Lagrange equation (4.3). Take a test function $h \in V_0^\infty(\mathbf{R}^n \setminus \{0\}$, that is h is supported in $\{x : |x| > \delta\}$ for some $\delta > 0$. Let also $0 < \epsilon << 1$ and consider $u = \phi + \epsilon h$. Recall $\int |x|^{-b} \phi^{p+1} dx = 1$. Since ϕ is a minimizer we should have

$$I_\omega[\phi + \epsilon h] \ge m(1) = N(\phi)$$

where $N(\phi) := \int |(-\Delta)^{s/2}\phi|^2 + \int \phi^2$ and $D(\phi) := \int |x|^{-b} (\phi)^{p+1} dx$. Thus,

$$\begin{aligned}
N(\phi + \epsilon h) &= \int |(-\Delta)^{s/2}(\phi + \epsilon h)|^2 + \int (\phi + \epsilon h)^2 \\
&= \int |(-\Delta)^{s/2}\phi + \epsilon(-\Delta)^{s/2} h|^2 + \int (\phi^2 + 2\epsilon h\phi + \epsilon^2 h^2) = \\
&= \int |(-\Delta)^{s/2}\phi|^2 + \int \phi^2 + 2\epsilon(\langle(-\Delta)^{s/2}\phi, (-\Delta)^{s/2} h\rangle + \langle h, \phi\rangle) + O(\epsilon^2) \\
&= N(\phi) + 2\epsilon\langle((-\Delta)^s + 1)\phi, h\rangle + O(\epsilon^2).
\end{aligned}$$

Similarly,

$$D(\phi + \epsilon h) = \int |x|^{-b} (\phi + \epsilon h)^{p+1} dx = 1 + (p+1)\epsilon\langle |x|^{-b}\phi^p, h\rangle + O(\epsilon^2)$$

It follows that

$$\begin{aligned} I_1(\phi+\epsilon h) &= \frac{N(\phi+\epsilon h)}{D[\phi+\epsilon h]^{\frac{2}{p+1}}} = \frac{N(\phi)+2\epsilon\langle((-\Delta)^s+1)\phi, h\rangle + O(\epsilon^2)}{1+2\epsilon\langle|x|^{-b}\phi^p, h\rangle + O(\epsilon^2)} = \\ &= N[\phi]+2\epsilon\langle((-\Delta)^s+1)\phi - |x|^{-b}N(\phi)\phi^p, h\rangle + O(\epsilon^2). \end{aligned}$$

As this holds for arbitrary function h and for all small ϵ, we have established that ϕ solves (4.3) in a distributional sense.

Finally, fix h to be a real-valued function, $h \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$. Starting again with the inequality

$$\frac{N(\phi+\epsilon h)}{D(\phi+\epsilon h)^{\frac{2}{p+1}}} \geq N(\phi),$$

but expanding to the second order[7] ϵ^2 , we obtain

$$N[\phi]+\epsilon^2[\langle \mathcal{L}_+ h, h\rangle + N[\phi](p+3)(\langle|\cdot|^{-b}\phi^p, h\rangle)^2] + O(\epsilon^3) \geq N[\phi],$$

after taking into account $\langle((-\Delta)^s+1)\phi - N(\phi)|x|^{-b}\phi^p, h\rangle = 0$. After taking limits as $\epsilon \to 0$, we derive

$$\langle \mathcal{L}_+ h, h\rangle \geq -N[\phi](p+3)(\langle|\cdot|^{-b}\phi^p, h\rangle)^2. \tag{4.7}$$

In particular, $\langle \mathcal{L}_+ h, h\rangle \geq 0$, if $\int |x|^{-b}\phi^p(x)h(x)dx = 0$. □

We shall now need to prove some further properties of the minimizers ϕ as well as some spectral results necessary for the sequel.

4.3. **Boundedness of ϕ.** In our next result, we use the already established (partial) coercivity of $\mathcal{L}_+$ on $\{|\cdot|^{-b}\phi^p\}^\perp \cap C_0^\infty(\mathbf{R}^n \setminus \{0\})$ in order to derive L^∞ bounds on ϕ. We believe that this is a new technique, which might be useful in the spectral analysis of other situations with singular potentials.

Once we show the boundedness of ϕ, we will go back to the claim about the coercivity of $\mathcal{L}_+$ on the full co-dimension one subspace $\{|\cdot|^{-b}\phi^p\}^\perp$.

Proposition 3. *Let $(n,s,p,b) \in \mathcal{A}$. Then, the minimizer ϕ constructed in Proposition 2 is a bounded function.*

Proof. Again, we assume $\omega = 1$, the other cases follow by rescaling.

We first show the boundedness of ϕ. Recall that since ϕ is a bell-shaped function, $\phi \in L^2(\mathbf{R}^n)$, we have that for every $x \neq 0$, $|\phi(x)| \leq C_n|x|^{-\frac{n}{2}}\|\phi\|_{L^2}$. This of course leaves the possibility that $\lim_{x\to 0}\phi(x) = \infty$, which we shall rule out for the remainder of the proof.

Our approach is by contradiction, that is assume that $\lim_{|x|\to 0}\phi(x) = \infty$. We now create a specifically designed test function $h \in C_0^\infty(\mathbf{R}^n \setminus \{0\}) \cup |x|^{-b}\phi^{p\perp}$. To this end, let χ be a radial positive C_0^∞ test function, supported in $\frac{1}{2} < |x| < 2$ and equal to 1 on $\frac{3}{4} < |x| < \frac{4}{3}$. Let $0 < \epsilon << 1$ and let

$$h(x) := \chi(x/\epsilon) - c_\epsilon\chi(x), \quad c_\epsilon = \frac{\int |x|^{-b}\phi^p(x)\chi(x/\epsilon)dx}{\int |x|^{-b}\phi^p(x)\chi(x)dx}$$

[7]Note that in the calculation above, the expansion in powers of ϵ is valid, since the fixed h that has its support away from zero

Clearly, $h \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$, where c_ϵ is designed so that $h \perp |x|^{-b}\phi^p(x)$. Note that the denominator of c_ϵ is bounded above and below by a constant independent on ϵ, so that

$$c_\epsilon \sim \int |x|^{-b}\phi^p(x)\chi(x/\epsilon)\,dx. \tag{4.8}$$

According to Proposition 2, we have that $\langle \mathscr{L}_+ h, h\rangle \geq 0$. As a consequence of this, after dropping some terms with favorable signs, we arrive at

$$c_\epsilon^2\langle(-\Delta)^s\chi,\chi\rangle - 2c_\epsilon\langle(-\Delta)^s\chi,\chi(\cdot/\epsilon)\rangle + \|(-\Delta)^{s/2}\chi(\cdot/\epsilon)\|^2 \geq pm(1)\int |x|^{-b}\phi^p(x)\chi^2(x/\epsilon)\,dx. \tag{4.9}$$

Let us estimate the terms on the left hand side of (4.9). Elementary estimates imply

$$\langle(-\Delta)^s\chi,\chi\rangle \leq C, \|(-\Delta)^{s/2}\chi(\cdot/\epsilon)\|^2 \leq C\epsilon^{n-2s}, c_\epsilon|\langle(-\Delta)^s\chi,\chi(\cdot/\epsilon)\rangle| \leq C\epsilon^n c_\epsilon,$$

The integral expression on the right hand side of (4.9) is essentially equivalent to c_ϵ, but not quite. In order to get the desired estimate, introduce the quantity $d_\epsilon := \int |x|^{-b}\phi^p(x)\chi^2(x/\epsilon)\,dx$, so that we now have proved the estimate

$$d_\epsilon \leq C(c_\epsilon^2 + \epsilon^{n-2s} + \epsilon^n c_\epsilon). \tag{4.10}$$

Furthermore, we have by Cauchy-Schwartz's inequality

$$c_\epsilon \leq C\int |x|^{-b}\phi^p(x)\chi(x/\epsilon)\,dx \leq C\left(\int |x|^{-b}\phi^p(x)\chi^2(x/\epsilon)\,dx\right)^{1/2}\left(\int_{|x|\sim\epsilon} |x|^{-b}\phi^p(x)\,dx\right)^{1/2}. \tag{4.11}$$

By our assumption, $\lim_{x\to 0}|\phi(x)| = \infty$, we have that for all small enough ϵ

$$\int_{|x|\sim\epsilon} |x|^{-b}\phi^p(x)\,dx \leq \frac{1}{\max_{x:|x|\sim\epsilon}\phi(x)}\int |x|^{-b}\phi^{p+1}(x)\,dx = \frac{1}{\max_{x:|x|\sim\epsilon}\phi(x)} = o(\epsilon).$$

Hence, we obtain that $c_\epsilon^2 = o(\epsilon)d_\epsilon$ and $\epsilon^n c_\epsilon \leq o(\epsilon)d_\epsilon + \epsilon^{2n}$. Substituting these estimates in (4.10) yields $d_\epsilon \leq Co(\epsilon)d_\epsilon + \epsilon^{n-2s}$, or after hiding $Co(\epsilon)d_\epsilon$ on the left-hand side, $d_\epsilon \leq 2\epsilon^{n-2s}$, for all small enough ϵ. This actually yields a very good point-wise estimate on ϕ. Indeed, recalling that ϕ is bell-shaped we estimate

$$c\epsilon^{n-b}\min_{x:|x|\sim\epsilon}\phi^p(x) \leq \int |x|^{-b}\phi^p(x)\chi^2(x/\epsilon)\,dx \leq Ce^{n-2s},$$

whence for all $x \neq 0$,

$$\phi^p(x) \leq C|x|^{b-2s}. \tag{4.12}$$

This gives a contradiction and hence the required L^∞ bound, if $b \geq 2s$. Unfortunately, this covers only a small portion of the parameters space $\mathscr{A}$.

So, assume for the rest of the argument that $b < 2s$. In order to derive the L^∞ bounds for ϕ, in the case $b < 2s$, we shall need an additional bootstrap argument, based on the fact that ϕ is a (weak) solution of the Euler-Lagrange equation (4.3). To this end, we need to find a way to introduce $\tilde{\phi} := (1+(-\Delta)^s)^{-1}[|\cdot|^{-b}\phi^p]$. As of now, this is a *formal definition*, but it is clear that if we manage to define such an object in an appropriate way, this will be weak solution of (4.3). Since ϕ solves (4.3) in the weak sense described in Proposition 2, we will be eventually able to show that $\tilde{\phi} = \phi$ as L^q functions, for appropriate $q \in (2,\infty)$. To this end, we have the following claim.

Claim 1. *Assume $(n,s,p,b) \in \mathscr{A}$ and that a function $f : \mathbf{R} \to \mathbf{R}$ is bell-shaped and it satisfies $f \in L^{p+1,-b}(\mathbf{R}^n)$ and $|f(x)| \le C|x|^{\frac{b-2s}{p}}$. Then,*

$$\tilde{z} = (1+(-\Delta)^s)^{-1}[|\cdot|^{-b} f^p] := G_s * [|\cdot|^{-b} f^p] \in \cap_{\frac{p+1}{p}<q} L^q(\mathbf{R}^n).$$

In particular $\tilde{z} \in L^2(\mathbf{R}^n)$.

Proof. (Claim 1) We consider the case $n > 2s$ only, as the case $n \le 2s$ can arise only for $n = 1$, $s > \frac{1}{2}$, in which case the function G_s is bounded and the arguments are much simpler.

We split[8] $\tilde{z} = \tilde{z}_1 + \tilde{z}_2$

$$\tilde{z}_1 = G_s * [|\cdot|^{-b} f^p \chi_{|\cdot|<1}], \quad \tilde{z}_2 = G_s * [|\cdot|^{-b} f^p \chi_{|\cdot|\ge 1}]$$

Let us analyze $\tilde{z}_1$ first. We claim that due to the properties established in Lemma 1, we have that $\tilde{z}_1 \in \cap_{q<\infty} L^q(\mathbf{R}^n)$. Indeed, for $|x| < 2$, we can bound

$$|\tilde{z}_1(x)| \le C|\cdot|^{2s-n}\chi_{|\cdot|<3} * |\cdot|^{-2s}\chi_{|\cdot|<1}.$$

Pick arbitrary $q_1, q_2 : 1 < q_1 < \frac{n}{n-2s}$, $1 < q_2 < \frac{n}{2s}$ and then $q \in (1,\infty) : \frac{1}{q_1} + \frac{1}{q_2} = 1 + \frac{1}{q}$. By Hardy-Littlewood-Sobolev inequality, we have

$$\|\tilde{z}_1\|_{L^q(|x|<2)} \le C\||\cdot|^{2s-n}\chi_{|\cdot|<3}\|_{L^{q_1}(\mathbf{R}^n)}\||y|^{-2s}\chi_{|\cdot|<1}\|_{L^{q_2}(\mathbf{R}^n)} \le C_q$$

Clearly, in this way, we can generate any $q \in (1,\infty)$, by varying the choices q_1, q_2 in the specified intervals, so $\tilde{z}_1 \in \cap_{1<q<\infty} L^q(\mathbf{R}^n)$.

Regarding $\tilde{z}_2$, we split as follows

$$|\tilde{z}_2| \le C[|\cdot|^{2s-n}\chi_{|\cdot|<1} * |\cdot|^{-b} f^p \chi_{|\cdot|\ge 1} + |\cdot|^{-n}\chi_{|\cdot|\ge 1} * |\cdot|^{-b} f^p \chi_{|\cdot|\ge 1}]$$

Clearly,

$$\||\cdot|^{2s-n}\chi_{|\cdot|<1} * |\cdot|^{-b} f^p \chi_{|\cdot|\ge 1}\|_{L^q} \le C\||\cdot|^{2s-n}\chi_{|\cdot|<1}\|_{L^1}\||\cdot|^{-b} f^p \chi_{|\cdot|\ge 1}\|_{L^q} \le C$$

as long as $\frac{p+1}{p} \le q < \infty$, because

$$\||\cdot|^{-b} f^p \chi_{|\cdot|\ge 1}\|_{L^q}^q \le \max_{|x|>1} |f^{qp-(p+1)}(x)| \int_{\mathbf{R}^n} |y|^{-b} f^{p+1}(y)\,dy \le C.$$

Similarly, as long as $\frac{p+1}{p} < q < \infty$, we can find $\delta > 0$, so that $\frac{1}{1+\delta} + \frac{1}{q_\delta} = 1 + \frac{1}{q}$ and $q_\delta > \frac{p+1}{p}$. Then,

$$\||\cdot|^{-n}\chi_{|\cdot|\ge 1} * |\cdot|^{-b} f^p \chi_{|\cdot|\ge 1}\|_{L^q} \le C\|\||\cdot|^{-n}\chi_{|\cdot|\ge 1}\|_{L^{1+\delta}}\||\cdot|^{-b} f^p \chi_{|\cdot|\ge 1}\|_{L^{q_\delta}} \le C.$$

All in all, we have established $\tilde{z} \in \cap_{\frac{p+1}{p}<q<\infty} L^q(\mathbf{R}^n)$, as required. □

Now that we have established the claim and taking into account the properties of ϕ, which are already established, we can take $f = \phi$ in the Claim 1, whence we conclude that

$$\tilde{\phi} = (1+(-\Delta)^s)^{-1}[|\cdot|^{-b}\phi^p]$$

is well-defined and element of $L^2(\mathbf{R}^n)$. Furthermore, for each integer k and each test function $f \in \mathscr{S}_k = \{f \in \mathscr{S} : supp\hat{f} \subset \{2^{k-1} \le |\xi| \le 2^{k+1}\}\}$, we have that

$$\langle \tilde{\phi}, (1+(-\Delta)^s)^{-1} f\rangle = \langle |\cdot|^{-b}\phi^p, f\rangle = \langle \phi, (1+(-\Delta)^s)^{-1} f\rangle,$$

where in the first equality we have used the definition of $\tilde{\phi}$, while in the second, we have used that ϕ is a weak solution of (4.3).

[8] here χ_I denotes the characteristic function of I

Since $(1+(-\Delta)^s)^{-1}$ is an isomorphism on each $\mathcal{S}_k$, it follows that $\langle\tilde{\phi}-\phi, f\rangle = 0$ for each $f \in \mathcal{S} : supp\hat{f} \subset \mathbf{R}^n \setminus \{0\}$. Since this is a dense set in $\mathcal{S}$ and hence in each $L^q, q \in [1,\infty)$, it follows that $\tilde{\phi} = \phi$ in the sense of $L^2(\mathbf{R}^n)$, that is

$$\phi = (1+(-\Delta)^s)^{-1}[|\cdot|^{-b}\phi^p] = G_s * [|\cdot|^{-b}\phi^p] \in L^2(\mathbf{R}^n). \tag{4.13}$$

According to the claim, the $L^2(\mathbf{R}^n)$ function on the right-hand side of (4.13) also belongs to $\cap_{\frac{p+1}{p}<q} L^q(\mathbf{R}^n)$. But then, since ϕ is bell-shaped and $\phi \in \cap_{\frac{p+1}{p}<q} L^q(\mathbf{R}^n)$, we have the point-wise bound

$$|x|^n |\phi(x)|^q \le C\int_{|y|\sim|x|} |\phi(y)|^q dy \le C_{q,n}\|\phi\|^q_{L^q(\mathbf{R}^n)}.$$

whence $\phi(x) \le C_q |x|^{-\frac{n}{q}}$. Recall that this is true for all $q<\infty$. That is, for each $\delta>0$, there is C_δ, so that

$$\phi(x) \le C_\delta |x|^{-\delta}. \tag{4.14}$$

This is almost, but not quite $\phi \in L^\infty(\mathbf{R}^n)$, which will yield the contradiction. On the other hand, we will show that (4.14) can be bootstrapped to $\phi \in L^\infty(\mathbf{R}^n)$, which will then be the desired contradiction.

By close inspection of the proof of Claim 1 (and under the assumptions in Claim 1) , we see that we can in fact place all but one piece in $L^\infty(\mathbf{R}^n)$. It thus remains to see why
$|\cdot|^{2s-n}\chi_{|\cdot|<3} * |\cdot|^{-b}\phi^p\chi_{|\cdot|<1} \in L^\infty(\mathbf{R}^n)$. In view of the bound (4.14), we have for $\delta << 1$,

$$|\cdot|^{2s-n}\chi_{|\cdot|<3} * |\cdot|^{-b}\phi^p\chi_{|\cdot|<1}(x)| \le C\int \frac{\chi_{|x-y|<3}}{|x-y|^{n-2s}}\frac{\chi_{|y|<1}}{|y|^{b+\delta}}dy \le C\||\cdot|^{2s-n}\chi_{|\cdot|<3}\|_{L^q}\|\chi_{|y|<1}|y|^{-b-\delta}\|_{L^r},$$

where in the last step, we have applied the Hölder's inequality with $1=\frac{1}{q}+\frac{1}{r}$, $q<\frac{n}{n-2s}$, $r(b+\delta)<n$. This last two conditions are possible to satisfy (i.e. such q,r exist) , for small δ, as long as $b<2s$. This is another instance that this requirement is crucially used. In this way, we have reached contradiction with our assumption that ϕ is unbounded. Therefore, ϕ is $L^\infty(\mathbf{R}^n)$ function. □

4.4. **Further properties of ϕ.** We have the following proposition.

Proposition 4. *Let $(n,s,p,b) \in \mathcal{A}$. Then, $\phi \in L^1(\mathbf{R}^n)$, so by the bell-shapedness, in particular it satisfies the point-wise bound*

$$|\phi(x)| \le C|x|^{-n}, |x|>1. \tag{4.15}$$

If in addition, $s \in (\frac{1}{2},1)$, then

$$|\nabla\phi(x)| \le C\begin{cases} |x|^{-n-1} & |x|>1 \\ |x|^{2s-b-1} & |x|<1 \end{cases} \tag{4.16}$$

In particular, $\phi \in C^1(\mathbf{R}^n \setminus \{0\})$.

Remarks: As a corollary, we have

- $\phi \in \cap_{1<q\le\infty} L^q(\mathbf{R}^n)$.
- $|x||\nabla\phi(x)|$ is a bounded function, since $2s>b$. In fact, $|x||\nabla\phi| \in \cap_{1<q\le\infty} L^q(\mathbf{R}^n)$.

Proof. Even though $\phi \in L^1$ implies (4.15), it will be actually bootstrapped from it. So, we focus on the proof of (4.15). We already know that $|\phi(x)| \le C|x|^{-n/2}, |x| > 1$. To obtain the higher decay rate, introduce the optimal decay rate,

$$\alpha := \sup\{s : |\phi(x)| \le A_s |x|^{-s}, |x| > 1\}.$$

Clearly $\alpha \ge \frac{n}{2}$. Assuming that $\alpha < n$ leads to a contradiction. Indeed, note the representation (4.13),

$$|\phi(x)| \le |G_s| * [|x|^{-b}\phi^p(x)|,$$

and the fact that G_s is integrable near zero. Moreover, there is the bound $|G_s(x)| \le C|x|^{-n}, |x| > 1$ and $|x|^{-n} * |x|^{-(b+p(\alpha-\epsilon))} \le C|x|^{-\min(n,b+p(\alpha-\epsilon))}$, for small enough ϵ, so that $b + p(\alpha-\epsilon) > \alpha$. But this implies a better decay rate than α. This contradicts our assumption $\alpha < n$, so it follows that $\alpha \ge n$. One can in fact see that $\alpha = n$, as this is the optimal decay rate for G_s.

The bound for $\|\phi\|_{L^1}$ follows easily now. We simply estimate

$$\|\phi\|_{L^1} \le \|G_s\|_{L^1} \||x|^{-b}\phi^p\|_{L^1} = \||x|^{-b}\phi^p\|_{L^1}.$$

But the function $|x|^{-b}\phi^p \sim |x|^{-b}, |x| < 1$, while $|x|^{-b}\phi^p \sim |x|^{-(b+np)}, |x| > 1$, so $|x|^{-b}\phi^p \in L^1(\mathbf{R}^n)$.

The bounds for $|\nabla\phi|$ for $|x| > 1$ follow as in the proof of (4.15), once we make sure that ∇G_s is integrable near zero, which since $|\nabla G_s(x)| \le C|x|^{2s-n-1}, |x| < 1$, requires that $s > \frac{1}{2}$. For the case $|\nabla\phi|, |x| < 1$, we again use the formula $\nabla\phi = \nabla G_s * [|\cdot|^{-b}\phi^p]$. One can see that for values $|x| < 1$,

$$|\nabla\phi(x)| \le C\int_{|y|<2} \frac{1}{|x-y|^{n+1-2s}} \frac{1}{|y|^b} dy + \text{bounded function.}$$

Integrating separately in the regions $|y| < \frac{|x|}{2}$ and $|y| \ge \frac{|x|}{2}$ yields the bound $|\nabla\phi(x)| \le C|x|^{2s-b-1}$. □

5. Preliminary spectral properties of $\mathcal{L}_\pm$

We start with the realization of $\mathcal{L}_\pm$ as a self-adjoint operator.

5.1. Self-adjointness of $\mathcal{L}_\pm$.

The conclusion $\phi \in L^\infty(\mathbf{R}^n)$ is helpful in our study of $\mathcal{L}_+$ and $\mathcal{L}_-$. However, we still face difficulties, for example with regards to the self-adjointness, as the potential $|x|^{-b}\phi^{p-1}(x)$ is still singular at zero. The following non-trivial lemma resolves these issues.

Lemma 5. *Let $(n, s, p, b) \in \mathcal{A}$ and in addition $2b < n$. Then the Friedrich's extensions of $\mathcal{L}_\pm$ are self-adjoint operators with the natural domain $H^{2s}(\mathbf{R}^n)$.*

Proof. Before we proceed with the construction of the Friedriech's extension, let us show that the condition $n > 2b$ ensures that $\mathcal{L}_\pm(H^{2s}) \subset L^2(\mathbf{R}^n)$. This reduces to the estimate

$$\left(\int_{\mathbf{R}^n} |x|^{-2b}|h(x)|^2 dx\right)^{1/2} \le C\|h\|_{H^{2s}(\mathbf{R}^n)},$$

which follows by (2.5), where $a = 2b$ and since $b < 2s$.

Next, introduce the quadratic forms $\mathcal{Q}_\pm[h,h] := \langle\mathcal{L}_\pm h, h\rangle$, with form domain $H^s(\mathbf{R}^n) \times H^s(\mathbf{R}^n)$. Via the usual Friedrich's procedure, it will suffice to show boundedness from below for $\mathcal{Q}_\pm$.

We proceed to bound $|\langle|x|^{-b}\phi^p, h\rangle|$. Clearly, the portion of the integral over $|x| > 1$ is easy to control,

$$\int_{|x|>1} |x|^{-b}\phi^p(x)|h(x)|dx \le C\|h\|_{L^2}\|\phi\|_{L^{2p}}^p \le C\|h\|_{L^2}.$$

For the piece over $|x|\leq 1$, we have by Cauchy-Schwartz and Sobolev embedding, for any[9] $\sigma: 0<\sigma<s, 2b<n+2\sigma$

$$|\int_{|x|\leq 1}|x|^{-b}\phi^p(x)h(x)dx|\leq \|(-\Delta)^{\frac{\sigma}{2}}h\|_Z L^2\|(-\Delta)^{-\frac{\sigma}{2}}[|x|^{-b}\phi^p\chi_{|x|\leq 1}]\|_{L^2}\leq$$
$$\leq C\|(-\Delta)^{\frac{\sigma}{2}}h\|_{L^2}\||x|^{-b}\chi_{|x|\leq 1}\|_{L^{\frac{2n}{n+2\sigma}}}\leq C\|(-\Delta)^{\frac{\sigma}{2}}h\|_{L^2}\leq \kappa\|(-\Delta)^{\frac{s}{2}}h\|_{L^2}+C_{\kappa,\sigma}\|h\|_{L^2}.$$

Next, for the integral $\int|x|^{-b}\phi^p h^2(x)dx$, we control it by applying Proposition 1, with $q=2$ and any $\sigma>\frac{b}{2}$,

$$\int|x|^{-b}\phi^p h^2(x)dx\leq C\|h\|^2_{H^\sigma}.$$

Choosing $\sigma<s$ as well, that is $\sigma\in(\frac{b}{2},s)$, we conclude that for each κ, there is C_κ, so that

$$\int|x|^{-b}\phi^p h^2(x)dx\leq \kappa\|h\|^2_{H^s}+C_\kappa\|h\|^2_{L^2}. \tag{5.1}$$

Combining the estimates for $\int|x|^{-b}\phi^p h dx$ and $\int|x|^{-b}\phi^p h^2(x)dx$, with (4.7), yields that there exists a sufficiently large C, so that for each $h\in H^s(\mathbf{R}^n)$, we have

$$\|(-\Delta)^{\frac{s}{2}}h\|^2_{L^2}-pm(\omega)\int|x|^{-b}\phi^p h^2(x)dx\geq -\kappa\|(-\Delta)^{\frac{s}{2}}h\|^2_{L^2}-C\|h\|^2_{L^2}.$$

or

$$(1+\kappa)\|(-\Delta)^{\frac{s}{2}}h\|^2_{L^2}-pm(\omega)\int|x|^{-b}\phi^p h^2(x)dx\geq -C\|h\|^2_{L^2}. \tag{5.2}$$

So, again by (5.1) and (5.2),

$$(1+\kappa)\|(-\Delta)^{\frac{s}{2}}h\|^2_{L^2}-2pm(\omega)\int|x|^{-b}\phi^p h^2(x)dx\geq -\kappa\|(-\Delta)^{\frac{s}{2}}h\|^2_{L^2}-C\|h\|^2_{L^2},$$

whence for small enough κ,

$$2(\|(-\Delta)^{\frac{s}{2}}h\|^2_{L^2}-pm(\omega)\int|x|^{-b}\phi^p h^2(x)dx)\geq -C\|h\|^2_{L^2},$$

which is the desired boundedness from below for $\mathscr{L}_+$, once we divide by two and add $\omega\|h\|^2_{L^2}$. Since $\mathscr{L}_-\geq\mathscr{L}_+$, the boundedness from below (and hence the self-adjointness of the Friedrich's extension) for $\mathscr{L}_-$ follows.

□

Corollary 2. *Under the assumption* $2b<n$, $\phi\in H^{2s}(\mathbf{R}^n)=D(\mathscr{L}_\pm)$.

Proof. Since $\phi\in L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$ is already clear, we just need to observe that $\phi=(1+(-\Delta)^s)^{-1}[|x|^{-b}\phi^p]\in\dot{H}^{2s}$. Indeed,

$$\|\phi\|_{\dot{H}^{2s}(\mathbf{R}^n)}=\|(-\Delta)^s(1+(-\Delta)^s)^{-1}[|x|^{-b}\phi^p]\|_{L^2}\leq C\||x|^{-b}\phi^p\|_{L^2},$$

which is finite, if $2b<n$ since $|x|^{-b}\phi^p\sim|x|^{-b}, |x|<1$ and for $|x|>1$, $|x|^{-b}\phi^p\leq\phi^p\in L^2(\mathbf{R}^n)$. □

Remark: The assumption $2b<n$ is automatic for $(n,p,s,b)\in\mathscr{A}$, if $n\geq 4$. In the case $n=3$ however, this is not so and it amounts to the extra restriction $b<\frac{3}{2}$. In [19], the authors use the fact that $\phi\in D(\mathscr{L}_\pm)$, which is not justified in the full range $n=3, b<2$, but rather only in the range $b<\frac{3}{2}$. Their statement has to be modified accordingly in order to hold, at least

[9]Clearly, one can select such $\sigma\in(0,s)$, as $b<n, b<2s$

based on the proof presented therein. Clearly, the restriction is even more severe in the lower dimensional cases $n = 1,2$.

Now that we have properly realized $\mathscr{L}_\pm$ as self-adjoint operators, one can talk about their eigenvalues, coercivity properties etc. Our next result are in this direction.

5.2. **Some basic coercivity properties of $\mathscr{L}_\pm$.**

Proposition 5. *Let $(n,s,p,b) \in \mathscr{A}$ and in addition $2b < n$. Then, the self-adjoint operators $\mathscr{L}_\pm$ enjoy the following properties:*

- *The continuous spectrum of $\mathscr{L}_\pm$ is $[\omega,\infty)$.*
- *$\mathscr{L}_+$ has exactly one negative eigenvalue.*
- *$\mathscr{L}_- \geq 0$, with $\mathscr{L}_-[\phi] = 0$ and moreover $\mathscr{L}_-|_{\{\phi\}^\perp\}} \geq 0$.*

Proof. Continuous spectrum for both operators consists of $[\omega,\infty)$ by Weyl's theorem. Clearly, since $\langle \mathscr{L}_+\phi,\phi\rangle = -(p-1)m(\omega)\int |x|^{-b}\phi^{p+1}dx < 0$, it follows that $\mathscr{L}_+$ has a negative eigenvalue. Then, the property $\langle \mathscr{L}_+ h, h\rangle \geq 0, h \perp |\cdot|^{-b}\phi^p$, which was previously established only for $h \in C^\infty(\mathbf{R}^n \setminus \{0\})$, can now be extended to all $h \in \mathscr{S} : h \perp |\cdot|^{-b}\phi^p$, since $|\cdot|^{-b}\phi^p \in L^2(\mathbf{R}^n)$, due to the assumption $2b < n$ and the properties of ϕ. Thus, $n(\mathscr{L}_+) = 1$.

Regarding the claims for $\mathscr{L}_-$, assume that the lowest eigenvalue, say $-\sigma^2$ is a negative one. Then,

$$-\sigma^2 = \inf_{\|u\|=1} \langle \mathscr{L}_- u, u\rangle = \inf_{\|u\|=1} [\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 + \omega - m(\omega)\int_{\mathbf{R}^n} |x|^{-b}\phi^p |u|^2 dx]$$

Similar to our considerations in the proof of Proposition 2, this variational problem has a bell-shaped solution, say $\psi : \|\psi\| = 1$, which satisfies $\mathscr{L}_-[\psi] = -\sigma^2\psi$. But on the other hand, by a direct inspection, $\mathscr{L}_-\phi = 0$, ϕ is bell-shaped as well. But then,

$$0 = \langle \mathscr{L}_-\phi,\psi\rangle = \langle \phi, \mathscr{L}_-\psi\rangle = -\sigma^2\langle\phi,\psi\rangle < 0,$$

a contradiction. Thus, $\mathscr{L}_-|_{\{\phi\}^\perp\}} \geq 0$.

□

Our next discussion will concern the Sturm-Liouville theory for fractional Schrödinger operators such as $\mathscr{L}_\pm$. We base our approach to a result due to Frank-Lenzmann-Silvester, [24].

5.3. **Sturm oscillation theorem for the second eigenfunction of $\mathscr{L}_+$.**

Theorem 4. *(Frank-Lenzmann-Silvestre, Theorem 2.3,* [24]*)*
Let $n \geq 1, s \in (0,1]$ and W satisfies

- *$W = W(|x|)$ and W is non-decreasing in $|x|$,*
- *$W \in L^\infty(\mathbf{R}^n)$, $W \in C^\gamma, \gamma > \max(0, 1-2s)$. That is*

$$|W(x) - W(y)| \leq C|x-y|^\gamma.$$

Then, assume that $H = (-\Delta)^s + W$ has two lowest radial eigenvalues E_0, E_1, so that $E_0 < E_1 < \inf\sigma_{ess}(H)$.

Then, the eigenvalue E_0 is simple and the corresponding eigenfunction is bell-shaped. Regarding E_1, the corresponding eigenfunction $\Psi_1 : \mathscr{H}\Psi_1 = E_1\Psi_1$ has exactly one change of sign. That is, there exists $r_0 \in (0,\infty)$, so that $\Psi_1(r) < 0, r \in (0,r_0)$ and $\Psi_1(r) > 0, r \in (r_0,\infty)$.

Remark: Note that the potentials involved in $\mathscr{L}_\pm$, while satisfying most of the requirements in Theorem 4, fail in a dramatic way the key boundedness requirement, as they are unbounded at zero. So, we shall need to employ an approximation argument to achieve the same result for $\mathscr{L}_+$.

Recall that according to Proposition 5, $\mathscr{L}_+$ has exactly one negative eigenvalue, $E_0 < 0$. The next *radial* eigenvalue E_1 (if there is one!) satisfies $E_1 \geq 0$.

Proposition 6. *(Sturm oscillation theorem for the second eigenfunction of $\mathscr{L}_+$)*
Let $(n, s, p, b) \in \mathscr{A}$ and in addition $2b < n$. Then, the smallest eigenvalue $E_0 < 0$ has a bell-shaped radial eigenfunction. Suppose that the operator $\mathscr{L}_+$ has a radial eigenvalue $E_1 < \omega$. Then, E_1 has a radial eigenfunction with exactly one change of sign.

Remark: The condition $E_1 < \omega$ simply means that E_1 is not an embedded eigenvalue, as $\sigma_{ac}(\mathscr{L}_+) = [\omega, \infty)$.

Proof. Before we start with the proof, let us mention that as we discuss radial eigenfunctions, we restrict our considerations to the Hilbert space $L^2_{rad}(\mathbf{R}^n)$ for the purposes of this proof.

Recall $\mathscr{L}_+ = (-\Delta)^s + \omega - pm(\omega)|x|^{-b}\phi^{p-1}(x) =: (-\Delta)^s + \omega - W$. The statements regarding E_0 can be established directly, even for the unbounded potential W. Indeed, by the self-adjointness of $\mathscr{L}_+$ and the characterization of the lowest eigenvalue

$$E_0 = \min_{\|u\|_{L^2}=1} \langle \mathscr{L}_+ u, u\rangle = \omega + \min_{\|u\|_{L^2}=1} [\|(-\Delta)^{\frac{s}{2}} u\|^2_{L^2} - \int_{\mathbf{R}^n} W(x)|u|^2 dx].$$

By the Polya-Szegö inequality and since $W = W^*$, $\int_{\mathbf{R}^n} W(x)|u|^2 dx \leq \int_{\mathbf{R}^n} W(x)|u^*|^2 dx$, we conclude that the minimization problem $\min_{\|u\|_{L^2}=1}\langle \mathscr{L}_+ u, u\rangle$ has a bell-shaped solution $\Psi_0 : \|\Psi_0\|_{L^2} = 1$ and $\mathscr{L}_+\Psi_0 = E_0\Psi_0$. In particular, $\Psi_0 \in H^{2s}(\mathbf{R}^n)$. Moreover, E_0 is a simple eigenvalue, as the minimizers for $\min_{\|u\|_{L^2}=1}\langle \mathscr{L}_+ u, u\rangle$ need to be bell-shaped and as such, cannot be orthogonal to Ψ_0.

Next, we define an approximation of W, namely for every integer N, the bounded potentials,

$$W_N(r) = \begin{cases} W(r) & r > \frac{1}{N} \\ W(N^{-1}) & r \leq \frac{1}{N} \end{cases}$$

and the operators $\mathscr{L}_{+,N} := (-\Delta)^s + \omega - W_N$. Note that $\mathscr{L}_{+,N} \geq \mathscr{L}_+$, since $W_N \leq W$.

As $W_N = W_N^*$, they have, by the same arguments as above ground states $\Psi_{0,N} : \|\Psi_{0,N}\|_{L^2} = 1$, corresponding to the smallest eigenvalues $E_{0,N} \geq E_0$, so $\mathscr{L}_{+,N}\Psi_{0,N} = E_{0,N}\Psi_{0,N}$. We will show that $\lim_N E_{0,N} = E_0$. Indeed, we have that

$$E_0 \leq E_{0,N} = \min_{\|u\|_{L^2}=1} \langle \mathscr{L}_{+,N} u, u\rangle \leq \langle \mathscr{L}_{+,N}\Psi_0, \Psi_0\rangle \leq E_0 + \int_{|x|<N^{-1}} W(|x|)\Psi_0^2(x) dx. \tag{5.3}$$

Since by (2.5), we have that

$$\left(\int_{|x|<1} |W(|x|)|\Psi_0^2(x) dx\right)^{1/2} \leq C\left(\int_{|x|<1} |x|^{-b}\Psi_0^2(x) dx\right)^{1/2} \leq C\|\Psi_0\|_{H^s(\mathbf{R}^n)}, \tag{5.4}$$

we conclude $\lim_{N\to\infty}\int_{|x|<N^{-1}} W(|x|)\Psi_0^2(x) dx = 0$, whence in combination with (5.3), we finally arrive at $\lim_N E_{0,N} = E_0$.

We now show that a subsequence of $\{\Psi_{0,N}\}$ converges strongly to Ψ_0. To that end, we need to show that $\{\Psi_{0,N}\}$ is pre-compact in the strong topology of $L^2(\mathbf{R}^n)$. Indeed, by (2.5), we have that, since $\frac{b}{2} < s$, there is C_s, so that

$$\int_{\mathbf{R}^n} W_N(|x|)\Psi_0^2 dx \leq C\int_{\mathbf{R}^n} |x|^{-b}\Psi_0^2 dx \leq C_s\|\Psi_0\|^2_{H^s(\mathbf{R}^n)}.$$

Thus, by Gagliardo-Nirenberg's inequality

$$E_{0,N} = \langle \mathscr{L}_{+,N}\Psi_{0,N}, \Psi_{0,N}\rangle \geq \|(-\Delta)^{\frac{s}{2}}\Psi_{0,N}\|_{L^2}^2 + \omega - C_s\|\Psi_0\|_{H^s(\mathbf{R}^n)}^2 \geq \frac{1}{2}\|(-\Delta)^{\frac{s}{2}}\Psi_{0,N}\|_{L^2}^2 - C_{s,\omega},$$

whence $\sup_N \|\Psi_{0,N}\|_{H^s} < \infty$. Next, by the representation $\Psi_{0,N} = ((-\Delta)^s + \omega - E_{0,N})^{-1}[W_N \Psi_{0,N}]$, $\|\Psi_{0,N}\|_{L^2} = 1$, and $\lim_N E_{0,N} = E_0 < 0$, we derive similar to the proof of (4.15), that there exists a constant $C = C_n$, but independent of N, so that $|\Psi_{0,N}(x)| \leq C_n |x|^{-n}$ for $|x| > 1$. This guarantees that $\lim_M \sup_N \int_{|x|>M} |\Psi_{0,N}(x)|^2 dx = 0$, which by Riesz-Relich-Kolmogorov criteria guarantees that $\{\Psi_{0,N}\}$ is pre-compact in $L^2(\mathbf{R}^n)$. That means that there is a subsequence $\Psi_{0,N_k} \to \Psi_0$. For simplicity of notations, we can assume without loss of generality that the sequence itself converges, i.e. $\lim_N \|\Psi_{0,N} - \Psi_0\|_{L^2} = 0$.

One can in fact show that (up to a further subsequence), $\lim_N \|\Psi_{0,N} - \Psi_0\|_{H^s} = 0$. Indeed, $\{\Psi_{0,N}\}$ being a bounded sequence in H^s has a weakly convergent subsequence (again assume that it is the sequence itself), which by uniqueness must be Ψ_0. Then, by lower semi-continuity of the L^2 norm with respect to weak convergence, $\liminf_N \|(-\Delta)^{\frac{s}{2}}\Psi_{0,N}\|_{L^2} \geq \|(-\Delta)^{\frac{s}{2}}\Psi_0\|_{L^2}$.

In addition, we claim that

$$\lim_N \int_{\mathbf{R}^n} W_N(|x|)\Psi_{0,N}^2(x)dx = \int_{\mathbf{R}^n} W(|x|)\Psi_0^2(x)dx. \tag{5.5}$$

Indeed, by (5.4), it suffices to show $\lim_N \left[\int_{\mathbf{R}^n} W_N(|x|)(\Psi_{0,N}^2(x) - \Psi_0^2(x))dx\right] = 0$. We have by Cauchy-Schwartz's that for every $\epsilon > 0$, there is C_ϵ

$$\begin{aligned}
&\left|\int_{\mathbf{R}^n} W_N(|x|)(\Psi_{0,N}^2(x) - \Psi_0^2(x))dx\right| \leq C\int_{\mathbf{R}^n} |x|^{-b}|\Psi_N(x) - \Psi_0(x)||\Psi_N(x) + \Psi_0(x)|dx \\
\leq\ & \left(\int_{\mathbf{R}^n} |x|^{-b}|\Psi_N(x) + \Psi_0(x)|^2\right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^n} |x|^{-b}|\Psi_N(x) - \Psi_0(x)|^2\right)^{\frac{1}{2}} \leq \\
\leq\ & C_\epsilon(\|\Psi_N\|_{H^s} + \|\Psi_0\|_{H^s})\|\Psi_N - \Psi_0\|_{H^{\frac{b}{2}+\epsilon}}.
\end{aligned}$$

where we have used (2.5). Note that by Gagliardo-Nirenberg's, we have

$$\|\Psi_N - \Psi_0\|_{H^{\frac{b}{2}+\epsilon}} \leq C\|\Psi_N - \Psi_0\|_{H^s}^{\frac{b/2+\epsilon}{s}} \|\Psi_N - \Psi_0\|_{L^2}^{\frac{s-b/2-\epsilon}{s}},$$

which clearly converges to zero, as $N \to \infty$, as long as we select $0 < \epsilon < s - b/2$.

Thus, having established (5.5) and $\liminf_N \|(-\Delta)^{\frac{s}{2}}\Psi_{0,N}\|_{L^2} \geq \|(-\Delta)^{\frac{s}{2}}\Psi_0\|_{L^2}$, we conclude

$$\begin{aligned}
E_0 &= \|(-\Delta)^{\frac{s}{2}}\Psi_0\|_{L^2}^2 + \omega - \int_{\mathbf{R}^n} W(|x|)\Psi_0^2(x)dx \leq \\
&\leq \liminf_N [\|(-\Delta)^{\frac{s}{2}}\Psi_{0,N}\|_{L^2}^2 + \omega - \int_{\mathbf{R}^n} W(|x|)\Psi_{0,N}^2(x)dx] = \liminf_N E_{0,N} = E_0.
\end{aligned}$$

It follows that $\liminf_N \|(-\Delta)^{\frac{s}{2}}\Psi_{0,N}\|_{L^2} = \|(-\Delta)^{\frac{s}{2}}\Psi_0\|_{L^2}$, which implies that (up to a subsequence) $\lim_N \|\Psi_{0,N} - \Psi_0\|_{H^s} = 0$.

We now turn to the second radial eigenfunction of $\mathscr{L}_+$. Let $h_1 \in D(\mathscr{L}_+) = H^{2s}(\mathbf{R}^n)$, $\|h_1\|_{L^2} = 1$ is an eigenfunction corresponding[10] to E_1, so $\mathscr{L}_+ h_1 = E_1 h_1$. Clearly $h_1 \perp \Psi_0$, whence $\lim_N \langle h_1, \Psi_{0,N}\rangle = 0$. By the Rayleigh characterization of the second smallest eigenvalue and since $\mathscr{L}_{+,N} \geq \mathscr{L}_+$, we have that $E_{1,N} \geq E_1$. Denote the corresponding *radial* eigenfunctions by $\Psi_{1,N} : \|\Psi_{1,N}\|_{L^2} = 1$. Note that $-W_N$ satisfy the requirements of Theorem 4, with $\gamma = 1$,

[10] Even though the ultimate claim is that there is an eigenfunction Ψ_1, which has exactly one change of sign, we do not know that yet

as a bounded, piecewise defined function, whose components are Lipschitz. Hence, due to Theorem 4, we may take those eigenfunctions $\Psi_{0,N}$ to have exactly one change of sign, say $r_N \in (0,\infty)$, say $\Psi_{0,N}|_{(0,r_N)} > 0, \Psi_{0,N}|_{(r_N,\infty)} < 0$.

Note

$$\begin{aligned} E_{1,N} &= \inf_{\|u\|_{L^2}=1, u\perp\Psi_{0,N}} \langle \mathcal{L}_{+,N} u, u\rangle \leq \frac{\langle \mathcal{L}_{+,N}(h_1 - \langle h_1, \Psi_{0,N}\rangle \Psi_{0,N}), h_1 - \langle h_1, \Psi_{0,N}\rangle \Psi_{0,N}\rangle}{\|h_1 - \langle h_1, \Psi_{0,N}\rangle \Psi_{0,N}\|^2} = \\ &= \langle \mathcal{L}_+ h_1, h_1\rangle + o(N^{-1}) = E_1 + o(N^{-1}). \end{aligned}$$

It follows that $\lim_N E_{1,N} = E_1$. In particular, the assumption $E_1 < \omega$ guarantees that[11] $E_{1,N} < \omega$ for large enough N. Similar to the proofs for $\Psi_{0,N}$, (in particular note the representation $\Psi_{1,N} = ((-\Delta)^s + \omega - E_{1,N})^{-1}[W_N \Psi_{1,N}]$, which implies the bound $|\Psi_{1,N}(x)| \leq C|x|^{-n}$ for $|x| > 1$), the system $\{\Psi_{1,N}\}$ is pre-compact in $L^2(\mathbf{R}^n)$, so it has a convergent subsequence. Again, assume that it is the sequence itself. Denote its limit by $\Psi_1 : \lim_N \|\Psi_{1,N} - \Psi_1\|_{L^2} = 0$.

Similar to the proof above for Ψ_0, we conclude that (after eventually taking a subsequence), $\lim_N \|\Psi_{1,N} - \Psi_1\|_{H^s} = 0$ and $\Psi_1 \perp \Psi_0$ is an eigenfunction for $\mathcal{L}_+$ corresponding to the eigenvalue E_1. It remains to show that Ψ_1 has exactly one sign change. To this end, consider the sequence $r_N \in (0,\infty)$ of sign changes for $\Psi_{1,N}$. There are three alternatives:

- $\{r_N\}$ converges to zero
- $\{r_N\}$ converges to $+\infty$
- $\{r_N\}$ has a subsequence, which converges to $r_0 \in (0,\infty)$.

We will show that the first two alternatives cannot really occur. Indeed, assume $r_N \to 0$. Then, pick a radial function $\zeta \in C_0^\infty(\mathbf{R}^n) : \zeta \geq 0$. We have

$$\langle \Psi_1, \zeta\rangle = \lim_N \langle \Psi_{1,N}, \zeta\rangle = \int_{|x|<r_N} \Psi_{1,N}\zeta(x)dx + \int_{|x|\geq r_N} \Psi_{1,N}\zeta(x)dx \leq 0$$

Thus, we conclude that $\Psi_1 \leq 0$ a.e., which is then a contradiction with $\langle \Psi_1, \Psi_0\rangle = 0$, as Ψ_0 is bell-shaped function. Similarly, the case $r_N \to \infty$ leads to the conclusion $\Psi_1 \geq 0$, which contradicts again $\Psi_1 \perp \Psi_0$.

Thus, the case $r_{N_k} \to r_0 > 0$ remains. For this subsequence, we clearly have that for each $\zeta : \zeta \in C_0^\infty(0, r_0), \zeta \geq 0$, we have $\langle \Psi_1, \zeta\rangle \geq 0$, while for $\zeta : \zeta \in C_0^\infty(r_0,\infty), \zeta \geq 0$, we have $\langle \Psi_1, \zeta\rangle \leq 0$. Equivalently, Ψ_0 changes sign exactly once, at $r_0 > 0$. □

6. THE NON-DEGENERACY OF Φ

In this section, we establish the non-degeneracy of the solutions of (1.2), obtained by means of rescaling of the constrained minimizers of (4.2). Let us outline the details of this construction. Start with a constrained minimizer ϕ_ω provided by Proposition 2. In particular, it satisfies (4.3), where recall $m(\omega)$ is in the form (4.1). Then, it suffices to take

$$\Phi_\omega(x) := m(\omega)^{\frac{1}{p-1}} \phi_\omega(x).$$

Clearly, with such a choice Φ_ω satisfies (1.2), which is bell-shaped and moreover enjoys all properties, as established for ϕ_ω in the Propositions 2, 3, 4. Note that $\mathcal{L}_\pm$ take the form

$$\mathcal{L}_+ = (-\Delta)^s + \omega - p|x|^{-b}\Phi_\omega^{p-1}, \mathcal{L}_- = (-\Delta)^s + \omega - |x|^{-b}\Phi_\omega^{p-1}.$$

The following result is the main conclusion of this section.

[11]And in fact, we may claim that $\omega - E_{1,N} \geq \frac{\omega - E_1}{2}$.

Proposition 7. *Assume $(n,p,s,b)\in\mathscr{A}$, and in addition $2b<n$ and $s\in(\frac{1}{2},1)$. Then,*

$$Ker[\mathscr{L}_+]=\{0\}.$$

We need to prepare the proof of Proposition 7 in several auxiliary results.

6.1. **Differentiation with respect to parameters.** We start this section with two *formal* calculations, which motivate our subsequent results.

6.1.1. *Taking formal derivatives.* Starting with the profile equation (1.2), we can *formally* take a derivative in any of the spatial variables, $\partial_{x_j}, j=1,\dots,n$. We obtain

$$\mathscr{L}_+[\partial_{x_j}\Phi]=-b\frac{x_j}{|x|^{b+2}}\Phi^p(x). \tag{6.1}$$

Let us emphasize again that (6.1) *is only a formal statement.* Indeed, such a formula is problematic at least in several ways - we need to have $\nabla\Phi\in D(\mathscr{L}_+)=H^{2s}$, the right-hand side of (6.1) is not in $L^2(\mathbf{R}^n)$, unless we assume $2(b+1)<n$ etc.

Similarly, by a simple scaling argument, the solution Φ_ω of (1.2) can be expressed through Φ_1, the solution for $\omega=1$ as follows

$$\Phi_\omega(x)=w^{\frac{2s-b}{2s(p-1)}}\Phi_1(\omega^{\frac{1}{2s}}x)=:\omega^{\sigma_p}\Phi_1(\omega^{\frac{1}{2s}}x). \tag{6.2}$$

This highlights the dependence on the parameter ω in (1.2), which will be very useful in the sequel. More specifically, the *formal* differentiation in ω yields

$$\mathscr{L}_+[\partial_\omega\Phi_\omega]=-\Phi_\omega. \tag{6.3}$$

Again, the formula (6.3) *is only a formal statement.* In particular, note that since $\partial_\omega\Phi_\omega$ can be expressed as a linear combination of Φ_ω and $x\cdot\nabla\Phi_\omega$, we have the same issues with respect to the domain of $\mathscr{L}_+$. In both instances, that is (6.1) and (6.3), we heuristically expect them to hold in some sense. The required technical tools, which establish the corresponding rigorous statementys, are developed next.

6.1.2. *A technical lemma.* The following lemma shows that one can take weak derivatives with respect to the spatial variables x as well as the parameter ω.

Lemma 6. *Let $q,\nabla q\in L^2(\mathbf{R}^n)$. Then, for any $\psi\in\mathscr{S}$,*

$$\lim_{\delta\to0}\langle\frac{q(x+\delta\mathbf{e}_j)-q(x)}{\delta},\psi\rangle=\langle\partial_{x_j}q,\psi\rangle, j=1,\dots,n, \tag{6.4}$$

Let now $q_\omega=f(\omega)q(g(\omega)x)$, where $f,g\in C^1(\mathbf{R}_+), g>0$ and $q,x\cdot\nabla_x q\in L^2(\mathbf{R}^n)$. Then, for any $\psi\in\mathscr{S}$, we have

$$\lim_{\delta\to0}\left\langle\frac{q_{\omega+\delta}-q_\omega}{\delta},\psi\right\rangle=\langle f'(\omega)q(g(\omega)\cdot)+f(\omega)g'(\omega)x\cdot\nabla_x q(g(\omega)\cdot),\psi\rangle. \tag{6.5}$$

Remark: Note that formally at least $\partial_\omega q=f'(\omega)q(g(\omega)\cdot)+f(\omega)g'(\omega)x\cdot\nabla_x q(g(\omega)\cdot)$, so the formula (6.5) is expected to be true.

Proof. We have by a simple change of variables

$$\lim_{\delta\to0}\langle\frac{q(x+\delta\mathbf{e}_j)-q(x)}{\delta},\psi\rangle=\lim_{\delta\to0}\langle q,\frac{\psi(\cdot-\delta\mathbf{e}_j)-\psi(\cdot)}{\delta}\rangle=-\langle q,\partial_j\psi\rangle=\langle\partial_j q,\psi\rangle,$$

where in the last step, we have used the Lebesgue's dominated convergence theorem integration by parts. This is justified since $\frac{\psi(\cdot-\delta\mathbf{e}_j)-\psi(\cdot)}{\delta}=-\partial_j\psi+O_{\|\cdot\|_{L^2}}(\delta)$ and $\nabla q\in L^2(\mathbf{R}^n)$. This establishes (6.4).

Regarding the proof of (6.5), by a change of variables and the Lebesgue's dominated convergence theorem

$$\lim_{\delta\to 0}\left\langle\frac{q_{\omega+\delta}-q_\omega}{\delta},\psi\right\rangle=\lim_{\delta\to 0}\int_{\mathbb{R}^n}\phi(y)\left(\frac{f(\omega+\delta)\psi(\frac{y}{g(\omega+\delta)})\frac{1}{g(\omega+\delta)^n}-f(\omega)\psi(\frac{y}{g(\omega)})\frac{1}{g(w)^n}}{\delta}\right)dy=$$
$$=\int_{\mathbb{R}^n}q(y)\partial_\omega\left[\frac{f(\omega)}{g(\omega)^n}\psi\left(\frac{y}{g(\omega)}\right)\right]dy=\left(\frac{f'(\omega)}{g^n(\omega)}-n\frac{f(\omega)g'(\omega)}{g^{n+1}(\omega)}\right)\int_{\mathbf{R}^n}q(y)\psi\left(\frac{y}{g(\omega)}\right)dy-$$
$$-\frac{f(\omega)g'(\omega)}{g^{n+2}(\omega)}\int_{\mathbf{R}^n}q(y)y\cdot\nabla_y\psi\left(\frac{y}{g(\omega)}\right)dy.$$

Clearly, the first term in (6.5) is accounted for as follows

$$\frac{f'(\omega)}{g^n(\omega)}\int_{\mathbf{R}^n}q(y)\psi\left(\frac{y}{g(\omega)}\right)dy=f'(\omega)\langle q(g(\omega)\cdot),\psi\rangle.$$

Next,

$$-n\frac{f(\omega)g'(\omega)}{g^{n+1}(\omega)}\int_{\mathbf{R}^n}q(y)\psi\left(\frac{y}{g(\omega)}\right)dy=-n\frac{f(\omega)g'(\omega)}{g(\omega)}\langle q(g(\omega)\cdot),\psi\rangle.$$

Finally, another change of variables and integration by parts (recall $q, x\cdot\nabla_x q\in L^2(\mathbf{R}^n)$ is assumed), yields

$$\int_{\mathbf{R}^n}q(y)y\cdot\nabla_y\psi\left(\frac{y}{g(\omega)}\right)dy=g^{n+1}(\omega)\int_{\mathbf{R}^n}q(g(\omega)x)x\cdot\nabla_x\psi(x)dx=$$
$$=-g^{n+1}(\omega)\int_{\mathbf{R}^n}div(xq(g(\omega)x))\psi(x)dx=-g^{n+1}(n\langle q(g(\omega)\cdot),\psi\rangle+g(\omega)\langle x\cdot\nabla_x q(g(\omega)\cdot),\psi\rangle).$$

Putting it all together yields the formula,

$$\lim_{\delta\to 0}\left\langle\frac{q_{\omega+\delta}-q_\omega}{\delta},\psi\right\rangle=f'(\omega)\langle q(g(\omega)\cdot),\psi\rangle+f(\omega)g'(\omega)\langle x\cdot\nabla_x q(g(\omega)\cdot),\psi\rangle$$

as required. □

Next, we have the following rigorous results which can be viewed as weaker versions of the formulas (6.1) and (6.3).

6.1.3. *Rigorous versions of the formal differentiation formulas.*

Proposition 8. *Let $(n,s,p,b)\in\mathcal{A}$, $s\in(\frac{1}{2},1), 2b<n$ and $\psi\in\mathcal{S}$. Then, any solution Φ_ω of (1.2), with the properties $\Phi\in L^2\cap L^\infty$ and $x\cdot\nabla\Phi\in L^2(\mathbf{R}^n)$ satisfies*

$$\langle\partial_j\Phi_\omega,\mathcal{L}_+\psi\rangle=-b\langle\frac{x_j}{|x|^{b+2}}\Phi^p,\psi\rangle,\ j=1,\dots,n \tag{6.6}$$
$$\langle\partial_\omega\Phi_\omega,\mathcal{L}_+\psi\rangle=-\langle\Phi_\omega,\psi\rangle, \tag{6.7}$$

Remarks:

- Note that the expression $\langle\frac{x_j}{|x|^{b+2}}\Phi^p,\psi\rangle$ is well-defined, for smooth functions ψ, whenever $2(b+1)<n$. This is however not always satisfied under the assumptions in Proposition

8. The expression still makes sense, under the weaker assumptions herein, provided we interpret it in the form

$$\langle \frac{x_j}{|x|^{b+2}}\Phi^p, \psi\rangle = \int_{\mathbf{R}^n} \frac{x_j}{|x|^{b+2}}\Phi^p(x)(\psi(x)-\psi(0))dx.$$

- The notation $\partial_\omega \Phi_\omega$ is used in (6.7) in the following sense

$$\partial_\omega \Phi_\omega = \sigma_p \omega^{\sigma_p - 1}\Phi_1(\omega^{\frac{1}{2s}}x) + \frac{\omega^{\sigma_p+\frac{1}{2s}-1}}{2s} x\cdot\nabla_x \Phi_1(\omega^{\frac{1}{2s}}x). \tag{6.8}$$

This is of course nothing but the formal derivative with respect to ω in (6.2). Note however that the expression on the right of (6.8) belongs to $L^2(\mathbf{R}^n)$, according to Proposition 4.

Proof. Our starting point is the formula (4.3). Applying it for x and $x+\delta\mathbf{e}_j$, taking the divided difference and then dot product with ψ yields

$$\langle ((-\Delta)^s+\omega)[\frac{\Phi(\cdot+\delta\mathbf{e}_j)-\Phi(\cdot)}{\delta}], \psi\rangle = \langle \frac{|\cdot+\delta|^{-b}\Phi^p(\cdot+\delta\mathbf{e}_j) - |\cdot|^{-b}\Phi^p(\cdot)}{\delta}, \psi\rangle. \tag{6.9}$$

Assume for the moment that ψ is so that $\hat{\psi}$ is supported in $\{\xi : |\xi|\geq \sigma>0\}$. In this way, $\tilde{\psi} = ((-\Delta)^s+\omega)\psi \in \mathscr{S}$, since its Fourier transform, $(\omega+(2\pi|\cdot|)^{2s})\hat{\psi}$ is in Schwartz class[12].

So we have, by (6.4),

$$\langle ((-\Delta)^s+\omega)[\frac{\Phi(\cdot+\delta\mathbf{e}_j)-\Phi(\cdot)}{\delta}], \psi\rangle = \langle \frac{\Phi(\cdot+\delta\mathbf{e}_j)-\Phi(\cdot)}{\delta}, \tilde{\psi}\rangle \to \langle \partial_j \Phi, \tilde{\psi}\rangle.$$

It follows that

$$\lim_{\delta\to 0}\langle ((-\Delta)^s+\omega)[\frac{\Phi(\cdot+\delta\mathbf{e}_j)-\Phi(\cdot)}{\delta}], \psi\rangle = \langle \partial_j\Phi, ((-\Delta)^s+\omega)\psi\rangle.$$

This clearly can be extended from the set of Schwartz functions, which are Fourier supported away from zero to the whole set $\mathscr{S}$. Indeed, it suffices to observe that the set of Schwartz functions, which are Fourier supported away from zero is H^{2s} dense in $\mathscr{S}$.

For the right-hand side of (6.9), we could perform an identical argument, except that we do not have in general that $\partial_j|\cdot|^{-b}\Phi^p(\cdot)\in L^2(\mathbf{R}^n)$ (as we would need to require $2(b+1)<n$). Instead, we proceed with the direct proof. We have

$$\langle \frac{|\cdot+\delta|^{-b}\Phi^p(\cdot+\delta\mathbf{e}_j) - |\cdot|^{-b}\Phi^p(\cdot)}{\delta}, \psi\rangle = \langle |\cdot|^{-b}\Phi^p(\cdot), \frac{\psi(\cdot-\delta\mathbf{e}_j)-\psi(\cdot)}{\delta}\rangle \to -\langle |\cdot|^{-b}\Phi^p(\cdot), \partial_j\psi\rangle.$$

If $\psi\in\mathscr{S}(\mathbf{R}^n\setminus\{0\})$, we can take integration by parts (as we avoid the singularity at zero), whence we arrive at

$$\lim_{\delta\to 0}\langle \frac{|\cdot+\delta|^{-b}\Phi^p(\cdot+\delta\mathbf{e}_j) - |\cdot|^{-b}\Phi^p(\cdot)}{\delta}, \psi\rangle = \langle -b\frac{x_j}{|x|^{b+2}}\Phi^p + p|x|^{-b}\Phi^{p-1}\Phi', \psi\rangle.$$

Again, one may extend such a formula from $\psi\in\mathscr{S}(\mathbf{R}^n\setminus\{0\})$ to $\psi\in\mathscr{S}$. It follows that taking limits as $\delta\to 0$ in (6.9) results in (6.6).

For the proof of (6.7), we proceed in a similar fashion. More specifically, taking (1.2) at ω and then at $\omega+\delta$ and subtracting yields the relation

$$((-\Delta)^s+\omega)[\frac{\Phi_{\omega+\delta}-\Phi_\omega}{\delta}] - |x|^{-b}[\frac{\Phi^p_{\omega+\delta}-\Phi^p_\omega}{\delta}] = -\Phi_{\omega+\delta}$$

[12] Note that $|\xi|^{2s}\hat{\psi}(\xi)$ is not smooth at zero, unless $\hat{\psi}$ vanishes in a neighborhood of zero

Taking dot product with $\psi \in \mathscr{S}(\mathbf{R}^n \setminus \{0\})$ yields

$$\langle \frac{\Phi_{\omega+\delta} - \Phi_\omega}{\delta}, ((-\Delta)^s + \omega))\psi\rangle - \langle |x|^{-b}[\frac{\Phi^p_{\omega+\delta} - \Phi^p_\omega}{\delta}], \psi\rangle = -\langle \Phi_{\omega+\delta}, \psi\rangle. \tag{6.10}$$

Clearly,

$$\langle \Phi_{\omega+\delta}, \psi\rangle = \langle \Phi_\omega, \psi\rangle + \delta\langle \frac{\Phi_{\omega+\delta} - \Phi_\omega}{\delta}, \psi\rangle \to \langle \Phi_\omega, \psi\rangle,$$

as the expression $\langle \frac{\Phi_{\omega+\delta}-\Phi_\omega}{\delta}, \psi\rangle$ has a limit by (6.5), namely $\langle \frac{\Phi_{\omega+\delta}-\Phi_\omega}{\delta}, \psi\rangle \to \langle \partial_\omega \Phi_\omega, \psi\rangle$.

Under the assumption $\psi \in \mathscr{S} : supp\hat{\psi} \subset \{\xi : |\xi| \geq \sigma > 0\}$, we introduce again $\tilde{\psi} = ((-\Delta)^s + \omega))\psi \in \mathscr{S}$. According to (6.2) and a simple change of variables

$$\lim_{\delta\to 0}\langle \frac{\Phi_{\omega+\delta} - \Phi_\omega}{\delta}, ((-\Delta)^s + \omega))\psi\rangle \quad = \quad \langle \partial_\omega \Phi_\omega, \tilde{\psi}\rangle = \langle \partial_\omega \Phi_\omega, ((-\Delta)^s + \omega))\psi\rangle.$$

This is again extendable, as above to any $\psi \in \mathscr{S}$. Finally, by (6.5) and the formula[13] $\partial_\omega \Phi^p_\omega = p\Phi^{p-1}_\omega \partial_\omega \Phi_\omega$, we have[14]

$$\lim_{\delta\to 0}\langle |\cdot|^{-b}[\frac{\Phi^p_{\omega+\delta} - \Phi^p_\omega}{\delta}], \psi\rangle = \lim_{\delta\to 0}\langle \frac{\Phi^p_{\omega+\delta} - \Phi^p_\omega}{\delta}, |\cdot|^{-b}\psi\rangle = p\langle \partial_\omega \Phi_\omega, |\cdot|^{-b}\Phi^{p-1}_\omega \psi\rangle.$$

All in all, we obtain (6.7).

□

6.2. **Spherical harmonics and fractional Schrödinger operators.** In this section, we give the final preparatory material before we establish the non-degeneracy, in the case $n \geq 2$. The approach is to decompose the fractional Schrödinger operator $\mathscr{L}_+ = (-\Delta)^s + \omega - p|x|^{-b}\Phi^{p-1}$, with a base space $L^2(\mathbf{R}^n)$ onto simpler, essentially one dimensional subspaces of the spherical harmonics (SH for short). This is convenient due to the radiality of the potential $W := p|x|^{-b}\Phi^{p-1}$, which allows for such decompositions to be invariant. In addition, the objects of interest are confined to the radial subspace and at most to the next SH subspace, which allows us to use Proposition 6. Similar approach was taken in the recent paper [47]. We continue now with the specifics.

The Laplacian on $\mathbb{R}^n$ is given in the spherical coordinates by

$$\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r + \frac{\Delta_{\mathbf{S}^{n-1}}}{r^2},$$

where $\Delta_{\mathbf{S}^{n-1}}$ is the self-adjoint Laplace-Beltrami operator on the sphere. Its action may be uniquely described as

$$\Delta_{\mathbf{S}^{n-1}} P[\vec{x}/r] = r^2 \Delta[P[\vec{x}/r]],$$

for each polynomial of n variables P. There are many useful properties of $\Delta_{\mathbf{S}^{n-1}}$, we will just concentrate the discussion on those that are directly relevant to our argument. In particular, its spectrum is explicitly given by

$$\sigma(-\Delta_{\mathbf{S}^{n-1}}) = \{l(l+n-2), l = 0, 1, ...\}$$

In fact, there are the finite dimensional eigenspaces $\mathscr{X}_l \subset L^2(\mathbf{S}^{n-1})$, corresponding to the eigenvalue $l(l+n-2)$, which give rise to the orthogonal decomposition $L^2(\mathbf{S}^{n-1}) = \oplus_{l=0}^\infty \mathscr{X}_l$. It is worth

[13] This formula is of course correct formally, but in order to provide a rigorous justification, we need to took into account (6.2), and (6.8)

[14] noting that $|\cdot|^{-b}\psi \in L^2(\mathbf{R}^n)$ under the standing assumption $2b < n$

noting that $\mathscr{X}_0 = span[1]$, whereas $\mathscr{X}_1 = span\{\frac{x_j}{r}, j = 1,2,...,n\}$. Denote $\mathscr{X}_{\geq 1} := \oplus_{l=1}^{\infty}\mathscr{X}_l$, so that $L^2(\mathbb{R}^n) = L^2_{rad}(r^{n-1}dr) \oplus L^2(r^{n-1}dr, \mathscr{X}_{\geq 1})$. We henceforth use the notation L^2_{rad} as a shorthand for $L^2_{rad}(r^{n-1}dr)$. Note that if we restrict $-\Delta$ to L^2_{rad} , we have

$$-\Delta|_{L^2_{rad}} = -\partial_{rr} - \frac{n-1}{r}\partial_r,$$

while

$$-\Delta|_{L^2(r^{n-1}dr,\mathscr{X}_{\geq 1})} \geq -\partial_{rr} - \frac{n-1}{r}\partial_r + \frac{n-1}{r^2}.$$

For every Banach space $X \hookrightarrow L^2(\mathbb{R}^n)$, we denote its radial subspace $X_{rad} := X \cap L^2_{rad}$.

Now consider a fractional Schrödinger operator $\mathscr{H} = (-\Delta)^s + W$, where W is radial. $\mathscr{H}$ acts invariantly on $L^2(r^{n-1}dr, \mathscr{X}_l)$ for each l. Upon introducing $\mathscr{H}_l = \mathscr{H}|_{L^2(r^{n-1}dr,\mathscr{X}_l)}$, we have the decomposition

$$\mathscr{H} = \oplus_{l=0}^{\infty}\mathscr{H}_l : \oplus_{l=0}^{\infty}L^2(r^{n-1}dr, \mathscr{X}_l) \to \oplus_{l=0}^{\infty}L^2(r^{n-1}dr, \mathscr{X}_l)$$

We also make use of the notation $\mathscr{H}_{\geq 1} := \oplus_{l=1}^{\infty}\mathscr{H}_l$ for $\mathscr{H}$ restricted to $\oplus_{l=1}^{\infty}L^2(r^{n-1}dr, \mathscr{X}_l)$. Clearly $D(\mathscr{H}_l) = D(\mathscr{H}) \cap L^2(r^{n-1}dr, \mathscr{X}_l)$ and $\sigma(\mathscr{H}) = \bigcup_{l=0}^{\infty}\sigma(\mathscr{H}_l)$ and $\mathscr{H}_0 < \mathscr{H}_1 < \mathscr{H}_2 < ....$ We shall also use the notation $\sigma_0(\mathscr{H}_l)$ for the bottom eigenvalue, $\sigma_1(\mathscr{H}_l)$ for the second smallest eigenvalue and so on.

6.3. **Conclusion of the non-degeneracy proof.** In this section, we follow the arguments in [47]. We also assume that $n \geq 2$, as the one dimensional case $n = 1$ reduces to an easy argument, contained in the proof below.

We have from Proposition 5 that $\mathscr{L}_+$ has one simple negative eigenvalue and from the previous section there is the decomposition of $\mathscr{L}_+$ in spherical harmonics as

$$\mathscr{L}_+ = \mathscr{L}_{+,0} \oplus \mathscr{L}_{+,\geq 1}.$$

The non-degeneracy of $\mathscr{L}_+$ follows from the following

Proposition 9. *$\sigma_1(\mathscr{L}_{+,0}) > 0$ and there exists $\delta > 0$ so that $\mathscr{L}_{+,\geq 1} \geq \delta > 0$*

Remark: We know that $\sigma_{ess.}(\mathscr{L}_+) = [\omega, \infty)$, whence the only remaining issue is the point spectrum.

Proof. We know that the smallest eigenvalue of $\mathscr{L}_+$, $E_0 < 0$ has a bell-shaped eigenfunction and hence, it is an eigenvalue of $\mathscr{L}_{+,0}$. The next *radial* eigenvalue E_1 cannot be negative since $n(\mathscr{L}_+) = 1$, thus $E_1 \geq 0$. If $E_1 > 0$, we will have shown $\sigma_1(\mathscr{L}_{+,0}) > 0$.

Assume, for a contradiction that $E_1 = 0$. Then by Proposition 6, there is an eigenfunction ψ_1 such that $\mathscr{L}_{+,0}\psi_1 = 0$, so that ψ_1 has exactly one change of sign. Without loss of generality, let $\psi_1(r) < 0, r \in (0, r_0)$ and $\psi_1(r) > 0$ for $r \in (r_0, \infty)$.

Next, we show now that $\Phi_\omega \perp Ker[\mathscr{L}_+]$. Indeed, for every $\psi \in ker[\mathscr{L}_+]$, we have that $\psi \in H^{2s}(\mathbf{R}^n)$. Thus, we can approximate by Schwartz functions $\psi_N \to \psi$ in $H^{2s}(\mathbf{R}^n)$ norm, whence $\lim_{N\to\infty} \|\mathscr{L}_+\psi_N - \mathscr{L}_+\psi\|_{L^2} = 0$. We have by (6.7) applied to ψ_N, that

$$0 = \langle \partial_\omega \Phi_\omega, \mathscr{L}_+\psi \rangle = \lim_{N\to\infty} \langle \partial_\omega \Phi_\omega, \mathscr{L}_+\psi_N \rangle = -\lim_{N\to\infty} \langle \Phi_\omega, \psi_N \rangle = -\langle \Phi_\omega, \psi \rangle.$$

It follows that $\Phi_\omega \perp Ker[\mathscr{L}_+]$. By a direct calculation we see that

$$\mathscr{L}_{+,0}\Phi = -|x|^{-b}(p-1)\Phi^p,$$

whence $|x|^{-b}\Phi^p \perp ker[\mathscr{L}_{+,0}]$. Note that since $2b < n$, $|x|^{-b}\Phi^p \in L^2(\mathbf{R}^n)$. Now consider

$$\varphi = c_0\Phi - r^{-b}\Phi^p = \Phi(c_0 - r^{-b}\Phi^{p-1}), c_0 := \frac{\Phi^{p-1}(r_0)}{r_0^b}.$$

Since Φ is bell-shaped, $\varphi(r) < 0, r \in (0, r_0)$ and $\varphi(r) > 0, r \in (r_0, \infty)$, but since $\varphi \perp ker[\mathscr{L}_{+,0}]$ we have $\langle\varphi, \psi_1\rangle = 0$. On the other hand, $\varphi\psi_1 \geq 0$, and this is a contradiction. Hence $\sigma_1(\mathscr{L}_{+,0}) > 0$.

Finally we show that $\mathscr{L}_{+,\geq 1} > 0$. Note however that since $n(\mathscr{L}_+) = 1$ and $n(\mathscr{L}_{+,0}) = 1$, we have $\mathscr{L}_{+,\geq 1} \geq 0$. Hence, we just need to show that zero is not eigenvalue for $\mathscr{L}_{+,\geq 1}$.

Suppose, for a contradiction, that zero is an eigenvalue for $\mathscr{L}_{+,\geq 1}$. This implies that zero is an eigenvalue for $\mathscr{L}_{+,1}$. Indeed, otherwise zero is then eigenvalue for $\mathscr{L}_{+,\geq 2}$, say $\mathscr{L}_{+,\geq 2}\vartheta = 0$. Since $\mathscr{L}_{+,\geq 2} > \mathscr{L}_{+,1}$, it will follow that

$$\langle\mathscr{L}_{+,1}\vartheta, \vartheta\rangle < \langle\mathscr{L}_{+,\geq 2}\vartheta, \vartheta\rangle = 0$$

Consequently, $\mathscr{L}_{+,1}$ has a negative eigenvalue, which is a contradiction, as we know $\mathscr{L}_{+,\geq 1} \geq 0$. Thus, we have reduced our contradiction argument to the case that $\mathscr{L}_{+,1}$ has an eigenvalue at zero, which we will need to refute now.

Since zero is now assumed to be an eigenvalue for $\mathscr{L}_{+,1}$ and $\mathscr{L}_{+,1} \geq 0$, it must be at the bottom of the spectrum. Its eigenfunctions are in the form $\psi_j = \psi(x)\frac{x_j}{|x|}$, $j = 1, \ldots, n$, where $\psi \in L^2_{rad}$. So, ψ is an eigenfunction at the bottom of the spectrum for the operator

$$\tilde{\mathscr{L}}_{+,1} = (-\partial_{rr} - \frac{n-1}{r}\partial_r + \frac{n-1}{r^2})^s + \omega - p|r|^{-b}\Phi^{p-1}(r),$$

acting on functions in L^2_{rad}. According to Lemma C.4, [24], $(-\Delta_l)^{\frac{s}{2}}, s \in (0,1)$ is positivity improving for each $l \geq 0$, i.e. for every $X_l \in \mathscr{X}_l$ and every $u \in \dot{H}^s_{rad}$,

$$\|(-\Delta_l)^{\frac{s}{2}}[uX_l]\|_{L^2_{rad}} \geq \|(-\Delta_l)^{\frac{s}{2}}|u|\|_{L^2_{rad}},$$

whence it is easy to see that $\langle\tilde{\mathscr{L}}_{+,1}u, u\rangle_{L^2_{rad}} \geq \langle\tilde{\mathscr{L}}_{+,1}|u|, |u|\rangle_{L^2_{rad}}$. Thus, we conclude that $\psi \geq 0$, since ψ is a solution of the constrained minimization problem

$$\begin{cases} \langle\tilde{\mathscr{L}}_{+,1}u, u\rangle_{L^2_{rad}} \to \min \\ \|u\|_{L^2_{rad}} = 1 \end{cases}$$

We now apply formula (6.6) for a sequence of Schwartz functions Ψ_N approximating $\psi_1(x) = \psi(x)\frac{x_1}{|x|} \in Ker[\mathscr{L}_+]$ in the $H^{2s}(\mathbf{R}^n)$ norm. We have

$$\begin{aligned} 0 &= \langle\partial_{x_1}\Phi, \mathscr{L}_+\psi_1\rangle = \lim_{N\to\infty}\langle\partial_{x_1}\Phi, \mathscr{L}_+\Psi_N\rangle = -b\lim_{N\to\infty}\langle\frac{x_1}{|x|^{b+2}}\Phi^p, \Psi_N\rangle = \\ &= -b\langle\frac{x_1}{|x|^{b+2}}\Phi^p, \psi_1\rangle = -b\int_{\mathbf{R}^n}\frac{x_1^2}{|x|^{b+3}}\Phi^p(x)\psi(x)dx < 0. \end{aligned}$$

which is a contradiction. Note that the last integral, the singularity at zero is integrable, since $b + 1 < n$, as $b < \frac{n}{2}, n \geq 2$. This concludes the proof of the proposition as well as the non-degeneracy of Φ. $\square$

7. Spectral and orbital stability of the waves

We start with some introductory material regarding the spectral stability of a general class of eigenvalue problems, of which ours will be a special case.

7.1. **Index counting theories: general theory.** We need a quick introduction of the instability index count theory, as developed in [37], [38], see also the book [39], as well as [45, 14, 40]. We will only consider special cases, which serve our purposes. To that end, we consider an eigenvalue problem in the form

$$\mathcal{J}\mathcal{L}f = \lambda f. \tag{7.1}$$

We need to introduce a a real Hilbert space, so that $f \in X$, its dual X^*, so that $\mathcal{L}: X \to X^*$, so that the bilinear form $(u,v) \to \langle \mathcal{L}u, v\rangle$ is a bounded symmetric bilinear form on $X \times X$. Next, $\mathcal{J}$ is assumed to be a bounded operator, which is skew-symmetric, i.e. $\mathcal{J}^* = -\mathcal{J}$. Furthermore, assume that there is an $\mathcal{L}$ invariant decomposition of the base space in the form

$$X = X_- \oplus Ker[\mathcal{L}] + \oplus X_+$$

where $\mathcal{L}|_{X_-} < 0$, $n(\mathcal{L}) := dim(X_-) < \infty$, $dim(Ker[\mathcal{L}]) < \infty$ and for some $\delta > 0$, $\mathcal{L}_{X_+} \geq \delta > 0$. That is, $\langle \mathcal{L}\Psi, \Psi\rangle \geq \delta \|\Psi\|_{X_+}$.

Next, consider the finite dimensional generalized eigenspace at the zero eigenvalue, defined as follows

$$E_0 = gKer[\mathcal{J}\mathcal{L}] = span[\cup_{k=1}^{\infty}[Ker[\mathcal{J}\mathcal{L}]^k]]$$

Note that $Ker[\mathcal{L}] \subset E_0$ and introduce $\tilde{E}_0 : E_0 = Ker[\mathcal{L}] \oplus \tilde{E}_0$. Consider the integer $k_0^{\leq 0}(\mathcal{L}) := n(\mathcal{L}|_{\tilde{E}_0})$. Equivalently, taking an arbitrary basis in $\tilde{E}_0$, $\{\psi_1, \ldots, \psi_N\} \subset D(\mathcal{L})$, define $k_0^{\leq 0}(\mathcal{L})$ to be the number of negative eigenvalues of the $N \times N$ matrix $\mathcal{D} = (\langle \mathcal{L}\psi_i, \psi_j\rangle)_{i,j,1\leq i,j\leq N}$.

Under these general assumptions, it is proved in [37] (see Theorem 1), that

$$k_r + 2k_c + 2k_0^{\leq 0} = n(\mathcal{L}) - n(\mathcal{D}), \tag{7.2}$$

where k_r is the number of real and positive solutions λ in (7.1), which account for the real unstable modes, $2k_c$ is the number of solutions λ in (7.1) with positive real part, which account for the modulational instabilities, and finally $2k_0^{\leq 0}$ is the number of the dimension of the marginally stable directions, corresponding to purely imaginary eigenvalue with negative Krein index.

7.2. **Index counting theory for** (1.5)**.** For the eigenvalue problem in the form (1.5), we have that $\mathcal{J}$ is invertible and anti-symmetric, $\mathcal{J}^{-1} = \mathcal{J}^* = -\mathcal{J}$ and $X = H^s(\mathbf{R}^n), X^* = H^{-s}(\mathbf{R}^n), n \geq 1$. Note that according to Proposition 5, we have that $n(\mathcal{L}_+) = 1$, while $n(\mathcal{L}_-) = 0$, whence $n(\mathcal{L}) = n(\mathcal{L}_+) + n(\mathcal{L}_-) = 1$. In addition,

$$Ker[\mathcal{L}] = span[\begin{pmatrix} ker[\mathcal{L}_+] \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ ker[\mathcal{L}_-] \end{pmatrix}] = span[\begin{pmatrix} 0 \\ \Phi_\omega \end{pmatrix}].$$

Thus, we have that $\mathcal{J}: Ker[\mathcal{L}] \to (Ker[\mathcal{L}])^\perp$. For the matrix $\mathcal{D}$, we need to solve $\Psi : \mathcal{J}\mathcal{L}\Psi = \begin{pmatrix} 0 \\ \Phi_\omega \end{pmatrix}$. So, $\Psi = \begin{pmatrix} \mathcal{L}_+^{-1}\Phi_\omega \\ 0 \end{pmatrix}$ and the matrix $\mathcal{D}$ is a scalar, with

$$\mathcal{D} = \langle \mathcal{L}\Psi, \Psi\rangle = \langle \mathcal{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle. \tag{7.3}$$

According to the formula (7.2), we conclude

$$k_r + 2k_c + 2k_0^{\leq 0} = 1 - n(\mathcal{D}).$$

Clearly, in our situation, it is always the case that $k_c = k_0^{\leq 0} = 0$, and $k_r = 1$ exactly when $\langle \mathcal{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle > 0$ and $k_r = 0$, when $\langle \mathcal{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle < 0$. We formulate our result in the following corollary.

Corollary 3. *For the eigenvalue problem* (1.5)*, spectral stability occurs exactly when* $\langle \mathscr{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle < 0$ *and instability is when* $\langle \mathscr{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle > 0$*. Moreover, the instability presents itself as a single, real unstable mode.*

Remarks:

- This is reminiscent of the standard Vakhitov-Kolokolov criteria for stability of waves in situations with a simple Morse index, i.e. Morse index equal to one.
- The case $\langle \mathscr{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle = 0$ presents a transition from stability to instability, so a pair of eigenvalues crosses from being purely imaginary $\pm\imath\sigma$ symmetric with respect to the origin to being a pair of real ones $\pm\lambda$. In this case, the algebraic multiplicity of the zero eigenvalue for $\mathscr{J}\mathscr{L}$ is four, up from the algebraic multiplicity two in all other cases, corresponding to the modulational invariance still present in the system.

7.3. **Coercivity of $\mathscr{L}_+$.** In this section show the coercivity property of $\mathscr{L}_+$ on the space $\{\Phi_\omega\}^\perp$.

Proposition 10. *Let* $(n,s,p,b)\in\mathscr{A}$ *and* $\langle \mathscr{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle < 0$*. Then, the operator* $\mathscr{L}_+$ *is coercive on* $\{\Phi_\omega\}^\perp \cap H^s$*. That is, there exists* $\delta > 0$*, so that for all*

$$\langle \mathscr{L}_+\Psi, \Psi\rangle \geq \delta \|\Psi\|_{H^s}^2, \quad \forall \Psi \perp \Phi_\omega. \tag{7.4}$$

Proof. This is a version of a well-known lemma in the theory, see for example Lemma 6.7 and Lemma 6.9 in [1]. Recall that we have already showed $Ker[\mathscr{L}_+] = \{0\}$ and $n(\mathscr{L}_+) = 1$. According to a result in [49] (see also Lemma 6.4, [1]), which state that under these conditions for $\mathscr{L}_+$

$$\alpha := \inf\{\langle \mathscr{L}_+ f, f\rangle : f \perp \Phi_\omega, \|f\|_{L^2} = 1\} \geq 0.$$

Consider the associated constrained minimization problem

$$\inf_{\|f\|=1, f\perp\Phi_\omega} \langle \mathscr{L}_+ f, f\rangle. \tag{7.5}$$

Take a minimizing sequence $f_k : \|f_k\| = 1, f_k \perp \Phi_\omega$, so that

$$\alpha = \lim_k \langle \mathscr{L}_+ f_k, f_k\rangle = \lim_k [\|(-\Delta)^{\frac{s}{2}} f_k\|^2 + \omega - p\int |x|^{-b}\Phi^{p-1}(x) f_k^2(x)dx].$$

By the properties

$$\|(-\Delta)^{\frac{s}{2}} f\| \geq \|(-\Delta)^{\frac{s}{2}} f^*\|, \quad \int |x|^{-b}\Phi^{p-1}(x) f^2(x)dx \leq \int |x|^{-b}\Phi^{p-1}(x)(f^*)^2(x)dx,$$

we can assume, without loss of generality that f_k are bell-shaped. Note that by (2.5) and the Gagliardo-Nirenberg's inequality

$$0 < \int |x|^{-b}\Phi^{p-1}(x) f_k^2(x)dx \leq C\|f_k\|^2_{H^{\frac{b}{2}+\epsilon}} \leq C\|f_k\|_{H^s}^{\frac{b/2+\epsilon}{s}} \|f_k\|_{L^2}^{\frac{s-b/2-\epsilon}{s}}.$$

Note that for $\epsilon = \frac{s-\frac{b}{2}}{2}$, by Young's inequality, we can derive the estimate (recall $\|f_k\|_{L^2} = 1$)

$$\langle \mathscr{L}_+ f_k, f_k\rangle \geq \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} f_k\|^2 - C_{n,s,b}.$$

It follows that $\sup_k \|(-\Delta)^{\frac{s}{2}} f_k\|^2 < \infty$. By bell-shapedness of $f_k : \|f_k\|_{L^2} = 1$, we have the pointwise bound $|f_k(x)| \leq C|x|^{-n/2}$. This, along with $\sup_k \|f_k\|_{H^s} < \infty$, easily implies compactness in any $L^q(|x| > 1), 2 < q < \infty$. On the other hand, in the bounded domain $|x| < 1$, there is compactness in $L^2(|x| < 1)$. So, assume without loss of generality that f_k itself converges to f strongly in all $L^q(|x| > 1), 2 < q < \infty$ and in $L^2(|x| < 1)$. In particular, f is bell-shaped, as f_k are bell-shaped. So, $f \neq 0$.

In addition to that, we can assume, without loss of generality a weak convergence in $H^s(\mathbf{R}^n)$, $f_k \rightharpoonup f$. Note that by the weak convergence,

$$f \perp \Phi_\omega, \ \liminf_k \|(-\Delta)^{\frac{s}{2}} f_k\|^2 \geq \|(-\Delta)^{\frac{s}{2}} f\|^2, \ \|f\|_{L^2} \leq \liminf \|f_k\|_{L^2} = 1.$$

Finally, by splitting in $|x| < 1$ and $|x| > 1$ and applying the different appropriate strong convergences in each (and uniform bounds in H^s), we obtain

$$\lim_k \int |x|^{-b} \Phi^{p-1}(x) f_k^2(x) dx = \lim_k \int |x|^{-b} \Phi^{p-1}(x) f^2(x) dx.$$

All in all, we obtain

$$\langle \mathscr{L}_+ f, f\rangle \leq \liminf \langle \mathscr{L}_+ f_k, f_k\rangle = \alpha. \tag{7.6}$$

We will now show that $\alpha > 0$. Assume for a contradiction that $\alpha = 0$. Since $f \neq 0$ (recall $f \perp \Phi_\omega$), we see from (7.6) that the function $g = \frac{f}{\|f\|}$ is a minimizer for (7.5). Writing the Euler-Lagrange equation for it implies

$$\mathscr{L}_+ g = \gamma g + c\Phi_\omega \tag{7.7}$$

Taking dot product with g and taking into account $\langle \mathscr{L}_+ g, g\rangle = 0, g \perp \Phi_\omega$ implies that $\gamma = 0$. This means that $g = c\mathscr{L}_+^{-1}\Phi_\omega$. But then,

$$0 = \langle \mathscr{L}_+ g, g\rangle = c^2 \langle \mathscr{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle.$$

Since $\langle \mathscr{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle \neq 0$ by assumption, it follows $c = 0$. But then, since $Ker[\mathscr{L}_+] = \{0\}$, (7.7) implies that $g = 0$, which is a contradiction.

So, we have shown that $\alpha > 0$. In other words,

$$\langle \mathscr{L}_+ \Psi, \Psi\rangle \geq \alpha \|\Psi\|^2, \ \forall \Psi \perp \Phi_\omega. \tag{7.8}$$

Note that (7.4) is however stronger than (7.8), as it involves $\|\cdot\|_{H^s}$ norms on the right-hand side. Nevertheless, we show that it is relatively straightforward to deduce it from (7.8). Indeed, assume for a contradiction in (7.4), that $g_k : \|g_k\|_{H^s} = 1, g_k \perp \Phi_\omega$, so that $\lim_k \langle \mathscr{L}_+ g_k, g_k\rangle = 0$.

Taking into account (7.8), this is only possible if $\lim_k \|g_k\|_{L^2} = 0$. So,

$$1 = \lim_k [\|(-\Delta)^{\frac{s}{2}} g_k\|_{L^2}^2 + \|g_k\|_{L^2}^2] = \lim_k \|(-\Delta)^{\frac{s}{2}} g_k\|_{L^2}^2.$$

But then, we achieve a contradiction

$$0 = \lim_k \langle \mathscr{L}_+ g_k, g_k\rangle = \lim_k [\|(-\Delta)^{\frac{s}{2}} g_k\|_{L^2}^2 + \omega\|g_k\|^2 - p\int |x|^{-b}\Phi^{p-1}(x) g_k^2(x) dx] = 1,$$

since $\lim_k \int |x|^{-b}\Phi^{p-1}(x) g_k^2(x) dx = 0$, similar to some previous steps, as $\sup_k \|(-\Delta)^{\frac{s}{2}} g_k\|_{L^2} < \infty$, $\|g_k\| \to 0$. A contradiction is reached, which completes the proof of Proposition 10.

□

Knowing that $\mathscr{L}_+|_{\{\Phi\}^\perp} \geq 0$ (and we have established something stronger in (7.4)), we can establishing the coercivity of $\mathscr{L}_-$.

7.4. **Coercivity of $\mathscr{L}_-$.** In Proposition 5, we have already established that $\mathscr{L}_-$ is non-negative on the subspace $\{\phi\}^\perp$. We need a stronger coercivity statement.

Proposition 11. *Let $(n,p,s,b) \in \mathscr{A}$. Then, there exists $\delta > 0$, so that*

$$\langle \mathscr{L}_- \Psi, \Psi \rangle \geq \delta \|\Psi\|^2_{H^s}, \forall \Psi \perp \Phi \tag{7.9}$$

Proof. Recall that in Proposition 6, we have already seen that $\mathscr{L}_-|_{\{\Phi\}^\perp} \geq 0$. We will show first that

$$\inf_{\|u\|=1, u\perp\phi} \langle \mathscr{L}_- u, u\rangle > 0.$$

Assuming not, it follows that $\mathscr{L}_-$ has a second eigenfunction in its kernel, $\tilde{\Phi} \perp \Phi$. But then, since $\mathscr{L}_+ < \mathscr{L}_-$, we have $\langle \mathscr{L}_+\tilde{\Phi}, \tilde{\Phi}\rangle < \langle \mathscr{L}_-\tilde{\Phi}, \tilde{\Phi}\rangle = 0$. Hence, $\mathscr{L}_+|_{\tilde{\Phi},\Phi\}^\perp} < 0$ and in particular, $\mathscr{L}_+$ has at least two negative eigenvalues, a contradiction. Thus, there exists $\delta > 0$, so that

$$\langle \mathscr{L}_- u, u\rangle \geq \delta \|u\|^2, u \perp \Phi \tag{7.10}$$

We would like to upgrade, as before, the right-hand side to $\|u\|^2_{H^s}$. To that end, we assume for a contradiction, that there is a sequence $u_k : u_k \perp \Phi, \|u_k\|_{H^s} = 1$, while $\lim_k \langle \mathscr{L}_- u_k, u_k\rangle = 0$. From (7.10), it follows that $\lim_k \|u_k\| = 0$, so $\lim_k \|(-\Delta)^{\frac{s}{2}} u_k\| = 1$. Similar to the proof of Proposition 10 above this yields a contradiction as well, since

$$0 = \lim_k \langle \mathscr{L}_- u_k, u_k\rangle = \lim_k [\|(-\Delta)^{\frac{s}{2}} u_k\|^2_{L^2} + \omega\|u_k\|^2 - \int |x|^{-b}\Phi^{p-1}(x) u_k^2(x)dx] = 1.$$

With this, (7.9) is established.

□

With Propositions 10 and 11 at hand, we are ready for the orbital stability result.

7.5. **Orbital stability of Φ_ω.** With the coercivity results in Proposition 10, one might argue that we have all the necessary ingredients for orbital stability, according to [33]. We are however missing one key piece of information, namely the map $\omega \to \Phi_\omega$ does not have the required C^1 smoothness. Therefore, we need a direct proof, which does not use the smoothness of this map.

Proposition 12. *Let the key assumptions (1), (2), (3) be satisfied and $\mathscr{L}_\pm|_{\{\Phi_\omega\}^\perp} \geq 0$, φ is non-degenerate, i.e $ker[\mathscr{L}_+] = \{0\}$, then $e^{-i\omega t}\Phi_\omega$ is orbitally stable solution of* (1.1).

Proof. Our proof proceeds by contradictions. More specifically, there is $\epsilon_0 > 0$ and a sequence of initial data $u_k : \lim_k \|u_k - \Phi\|_{H^s(\mathbb{R}^n)} = 0$, so that

$$\sup_{0\leq t<\infty} \inf_{\theta\in\mathbf{R}} \|u_k(t,\cdot) - e^{i\theta}\Phi\|_{H^s} \geq \epsilon_0.$$

Recall that $E[u] = \mathscr{H}[u] + \frac{w}{2}\mathscr{P}[u]$. Introduce

$$\epsilon_k := |E[u_k(t)] - E[\Phi_\omega]]| + |\mathscr{P}[u_k(t)] - \mathscr{P}[\Phi_\omega]]|.$$

Since we have assumed the conservation laws, we have that ϵ_k is conserved and $\lim_k \epsilon_k = 0$ For all $\epsilon > 0$, define

$$t_k = \sup\{\tau : \sup_{0<t<\tau} \|u_k(t) - \Phi\|_{H^s(\mathbb{R}^n)} < \epsilon\}$$

Note that $t_k > 0$, by the local well-posedness assumption (1). If we let $u_k = v_n + i w_k$, then for $t \in (0, t_k)$, we have $\|w_k(t)\|_{H^s(\mathbb{R}^n)} \leq \|u_k(t) - \Phi\|_{H^s(\mathbb{R}^n)} < \epsilon$. Define the modulations parameter $\theta_k(t)$ so that $[w_k(t) - \sin(\theta_k(t))\Phi] \perp \Phi$, which is

$$\sin(\theta_k(t))\|\Phi\| = \langle w_k(t), \Phi\rangle \tag{7.11}$$

Since $|\langle w_k(t),\Phi\rangle|\leq \epsilon\|\Phi\|_{L^2}$, there is an unique small solution $\theta_k(t)$ of 7.11, with $|\theta_k(t)|\leq\epsilon$. In addition, we have

$$\|u_k(t,\cdot)-e^{i\theta_k(t)}\varphi\|_{H^s}\leq \|u_k(t,\cdot)-\Phi\|_{H^s}+|e^{i\theta_k(t)}-1|\|\Phi\|_{H^s}\leq C_0\epsilon,$$

where $C_0=C_0(\|\Phi\|_{H^s})$ only. Let

$$T_k=\sup\{\tau:\sup_{0<t<\tau}\|u_k(t)-e^{i\theta_k(t)}\varphi(.)\|_{H^s(\mathbb{R}^n)}<2C_0\epsilon\}$$

Clearly $T_k>t_k>0$ and to complete the proof it is enough to show that for all $\epsilon>0$ and large k $T_k=\infty$, since we can choose $\epsilon_k:\epsilon_k<<\epsilon_0$.

For $t\in(0,T_k)$, write

$$\psi_k(t,.)=u_k(t,\cdot)-e^{i\theta_k(t)}\Phi$$

and decompose into real and imaginary parts of ψ_k and then project on the vector $\begin{pmatrix}\Phi\\0\end{pmatrix}$. This yields

$$\begin{pmatrix} v_n(t,\cdot)-\cos(\theta_k(t))\Phi\\ w_k(t,\cdot)-\sin(\theta_k(t))\Phi\end{pmatrix}=\mu_k(t)\begin{pmatrix}\Phi\\0\end{pmatrix}+\begin{pmatrix}\eta_k(t,\cdot)\\ \zeta_k(t,\cdot)\end{pmatrix},\ \begin{pmatrix}\eta_k(t,\cdot)\\ \zeta_k(t,\cdot)\end{pmatrix}\perp\begin{pmatrix}\Phi\\0\end{pmatrix} \tag{7.12}$$

Note that this decomposition implies $\eta_k(t)\perp\Phi$, while $\zeta_k(t)=w_k(t,\cdot)-\sin(\theta_k(t))\Phi\perp\Phi$ by the choice of θ_k, see (7.11). Taking L^2 norms in (7.12) yields

$$|\mu_k(t)|^2\|\Phi\|^2+\|\eta_k(t)\|^2+\|\zeta_k(t)\|^2=\|\psi_k(t)\|^2\leq 4C_0^2\epsilon^2. \tag{7.13}$$

We now exploit the properties of the conserved quantities. We have

$$\mathscr{P}[u_k(t)] \;=\; \int_{\mathbf{R}^n}|e^{i\theta_k(t)}\Phi+\psi_k(t)|^2dx=\mathscr{P}[\Phi]+\|\psi_k(t,\cdot)\|_{L^2}^2+2\int_{\mathbf{R}^n}\Phi(x)\Re[e^{i\theta_k(t)}\psi_k(t,x)]dx.$$

But

$$\begin{aligned}\int\Phi(x)\Re[e^{i\theta_k(t)}\psi_k(t,x)]dx &= \int\Phi(x)[\cos(\theta_k)(v_n-\cos(\theta_k)\Phi)-\sin(\theta_k)(w_k-\sin(\theta_k)\Phi)]dx=\\ &= \mu_k(t)\cos(\theta_k(t))\|\phi\|^2,\end{aligned}$$

due to $\eta_k\perp\Phi$ and $w_k-\sin(\theta_k)\Phi\perp\Phi$.

It follows that,

$$\mathscr{P}[u_k(t)]=\mathscr{P}[\Phi]+\|\psi_k(t,\cdot)\|_{L^2}^2+2\mu_k(t)\cos(\theta_k(t))\|\Phi\|^2,$$

whence by recalling that $\|\psi_k(t,\cdot)\|_{L^2}\leq 2C_0\epsilon$, in $t:0<t<T_k$

$$|\mu_k(t)|\leq\frac{|\mathscr{P}[u_k(t)]-\mathscr{P}[\phi]|+\|\psi_k(t,\cdot)\|_{L^2}^2}{2\cos(\theta_k(t)\|\Phi\|^2}\leq C(\epsilon_k+\|\psi_k(t,\cdot)\|_{L^2}^2)\leq C(\epsilon_k+\epsilon^2). \tag{7.14}$$

In the last estimate, recall that $|\theta_k(t)|\leq C_0\epsilon<<1$, whence $\cos(\theta_k(t))\geq\frac{1}{2}$ and the denominator is harmless.

Next, we take advantage of an expansion for $E[u_k(t)]-E[\Phi]$. Indeed, for all sufficiently small ϵ, we have

$$E[u_k(t)]-E[\Phi]=E[e^{i\theta_k(t)}\Phi+\psi_k]-E[\Phi]=E[\Phi+e^{-i\theta_k(t)}\psi_k]-E[\Phi].$$

Generally, for small perturbations of the wave $\varrho_1+i\varrho_2\in H^s(\mathbf{R}^n)$ and by taking into account the specific form of the energy functional E, we have

$$E[\Phi+(\varrho_1+i\varrho_2)]-E[\Phi]=\frac{1}{2}[\langle\mathscr{L}_+\varrho_1,\varrho_1\rangle+\langle\mathscr{L}_-\varrho_2,\varrho_2\rangle]+Err[\varrho_1,\varrho_2], \tag{7.15}$$

where

$$|Err[\varrho_1,\varrho_2]| \le C\int_{\mathbf{R}^n}|x|^{-b}\left||\Phi+\varrho_1+i\varrho_2|^{p+1}-\Phi^{p+1}-(p+1)\Phi^p\varrho_1-\frac{p(p+1)}{2}\varrho_1^2-\frac{p+1}{2}\varrho_2^2\right|dx.$$

Observe that by elementary second order Taylor expansions of the function $z\to|z|^{p+1}$, there is the pointwise estimate

$$\left||\Phi+\varrho_1+i\varrho_2|^{p+1}-\Phi^{p+1}-(p+1)\Phi^p\varrho_1-\frac{p(p+1)}{2}\varrho_1^2-\frac{p+1}{2}\varrho_2^2\right|\le C(\|\Phi\|_{L^\infty})(|\varrho_1|+|\varrho_2|)^{\min(p+1,3)},$$

whence, according to (2.4), we obtain the estimate

$$|Err[\varrho_1,\varrho_2]|\le C\int_{\mathbf{R}^n}|x|^{-b}(|\varrho_1|^{\min(p+1,3)}+|\varrho_2|^{\min(p+1,3)})dx\le C(\|\varrho_1\|_{H^s}^{\min(p+1,3)}+\|\varrho_2\|_{H^s}^{\min(p+1,3)}).$$

Apply this expansion (7.15) to

$$\varrho_1+i\varrho_2=e^{-i\theta_k(t)}\psi_k=\left[\cos(\theta_k)(\mu_k\Phi+\eta_k)+\sin(\theta_k)\zeta_k\right]+i\left[\cos(\theta_k)\zeta_k-\sin(\theta_k)(\mu_k\Phi+\eta_k)\right].$$

From (7.13), we see that $\|\varrho_1\|_{H^s}+\|\varrho_2\|_{H^s}\le C\epsilon$, so we can bound the contribution of $|Err[\varrho_1,\varrho_2]|$ as follows

$$|Err[\varrho_1,\varrho_2]|\le C\epsilon^{\min(p-1),1}(\|\varrho_1\|_{H^s}^2+\|\varrho_2\|_{H^s}^2). \tag{7.16}$$

Furthermore,

$$\langle\mathscr{L}_+\varrho_1,\varrho_1\rangle=\langle\mathscr{L}_-\eta_k,\eta_k\rangle-C(\epsilon^3+\epsilon_k+\epsilon^2(\|\eta_k\|_{H^s}+\|\zeta_k\|_{H^s})+\epsilon(\|\eta_k\|_{H^s}+\|\zeta_k\|_{H^s})^2)$$
$$\langle\mathscr{L}_-\varrho_2,\varrho_2\rangle\ge\langle\mathscr{L}_-\zeta_k,\zeta_k\rangle-C(\epsilon^3+\epsilon_k+\epsilon^2(\|\eta_k\|_{H^s}+\|\zeta_k\|_{H^s})+\epsilon(\|\eta_k\|_{H^s}+\|\zeta_k\|_{H^s})^2)$$

Due to the coercivity of $\mathscr{L}_-$ (see Proposition 11 and more specifically 7.9) and $\mathscr{L}_+$, which was established in Proposition 10, we have that for some $\kappa>0$ and since $\eta_k,\zeta_k\perp\Phi$, we have

$$\begin{aligned}\epsilon_k &\ge |E[u_k(t)]-E[\Phi]|\ge\\ &\ge \kappa(\|\eta_k\|_{H^s}^2+\|\zeta_k\|_{H^s}^2)-C(\epsilon^3+\epsilon_k+\epsilon^2(\|\eta_k\|_{H^s}+\|\zeta_k\|_{H^s})+\epsilon^{\min(p-1),1}(\|\eta_k\|_{H^s}+\|\zeta_k\|_{H^s})^2),\end{aligned}$$

or in other words, after some algebraic manipulations and for sufficiently small ϵ (depending only on absolute constant),

$$\|\eta_k(t)\|_{H^s}^2+\|\zeta_k(t)\|_{H^s}^2\le C(\epsilon^3+\epsilon_k), \tag{7.17}$$

where C is a constant that depends on the parameters, but not on ϵ and n. We claim that this implies that $T_k^*=\infty$ for sufficiently small ϵ (depending on the parameters only) and then sufficiently large k, so that $\epsilon_k<<\epsilon$. Indeed, assume that $T_k^*<\infty$. Then

$$2C_0\epsilon=\limsup_{t\to T_k^*-}\|\psi_k(t)\|_{H^s}\le C(|\mu_k(t)|+\|\eta_k(t)\|_{H^s}+\|\zeta_k(t)\|_{H^s})\le C(\epsilon^{\frac{3}{2}}+\sqrt{\epsilon_k}).$$

This last inequality is a contradiction, if $\epsilon: C_0\epsilon\ge C\epsilon^{\frac{3}{2}}$ and then $C\sqrt{\epsilon_k}<C_0\epsilon$. Both of this can be arranged, so we obtain the required contradiction, which establishes Proposition 12. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, 1460 JAYHAWK BOULEVARD, LAWRENCE KS 66045–7523, USA
Email address: aramadan@ku.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, 1460 JAYHAWK BOULEVARD, LAWRENCE KS 66045–7523, USA
Email address: stefanov@ku.edu