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ON THE SPECTRAL PROBLEM $\mathcal{L}u = \lambda u'$ AND APPLICATIONS

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Abstract. We develop a general instability index theory for an eigenvalue problem of the type $\mathcal{L}u = \lambda u'$, for a class of self-adjoint operators $\mathcal{L}$ on the line $\mathbf{R}^1$. More precisely, we construct an Evans-like function to show (a real eigenvalue) instability in terms of a Vakhitov-Kolokolov type condition on the wave. If this condition fails, we show by means of Lyapunov-Schmidt reduction arguments and the Kapitula-Kevrekidis-Sandstede index theory that spectral stability holds. Thus, we have a complete spectral picture, under fairly general assumptions on $\mathcal{L}$. We apply the theory to a wide variety of examples. For the generalized Bullough-Dodd-Tzitzeica type models, we give instability results for travelling waves. For the generalized short pulse/Ostrovsky/Vakhnenko model, we construct (almost) explicit peakon solutions, which are found to be unstable, for all values of the parameters.

1. Introduction

In our considerations below, the main motivation model is given by

$$u_{tx} = au - f(u). \tag{1}$$

Here $a > 0$ and f is a smooth function of u, so that $f(u) = O(u^2), f'(u) = O(u)$ for small u. Representative examples of actual physical/geometrical models of this type, and we will refer to them as generalized Bullough-Dodd equations, are provided in the following (incomplete) list ([3, 8, 19, 30]

$$u_{tx} = e^u - e^{-2u} \tag{2}$$
$$u_{tx} = e^u - e^{-2u} \tag{3}$$
$$u_{tx} = \sinh(u) \tag{4}$$
$$u_{tx} = e^u \tag{5}$$

Here, (2) is often referred to as the Tzitzeica equation and also the Bullough-Dodd equation[1], (3) is the related Tzitzeica-Bullough-Dodd model, (4) is the sinh-Gordon model, while (5) is the Liouville equation. These models exhibit explicit travelling wave solutions ([19, 30]), which are unfortunately unbounded on $\mathbf{R}^1$.

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[1] The derivation by Tzitzeica was related to problems in classical differential geometry, while Bullough-Dodd, [3] have derived in the context of the Klein-Gordon equation

Another example with similar structure is the generalized short pulse equation. More precisely, we refer to

$$u_{tx} = u + (u^p)_{xx}. \tag{6}$$

For the case $p = 2$, this is referred to as Ostrovsky equation in [16], but is also referred to as reduced Ostrovsky [17, 27], short wave equation [10], Ostrovsky-Hunter equation, [1] etc. This equation arises in different settings, for example as a model for small amplitude long waves in rotating fluids. The model with $p = 3$ on the other hand has found nonlinear optics applications as a model for very short pulse propagation in non-linear media, [23], hence our adoption of the name short pulse for the whole hierarchy. Many works have explored various aspects of local and global well-posedness. Short time solutions were shown to exist, when the data is in high enough order Sobolev spaces, [23], [26]. On the other hand, solutions to (6) exist globally for small data and generically exhibit finite time blow up for large data, [9].

In this article, we are interested in the existence of travelling wave solutions for these models and their stability properties. It turns out that one can develop a pretty general instability index theory that treats the relevant eigenvalue problems. In fact, this theory is able to handle the instability properties of waves arising in models, such as the Bullough-Dodd, Tzitzeica, the Liouville equation etc. See Section 4 for the precise definitions and instability results as well as Section 5 for results about peakons for the short pulse equation.

To fix ideas, we consider spectral problems of the form[2]

$$\mathcal{L}u = \lambda u', \tag{7}$$

where $\mathcal{L}$ is a self-adjoint operator, λ is a complex number and u is a function belonging to $D(\mathcal{L}) = H^s(\mathbf{R}^1) \subset H^1(\mathbf{R}^1)$. Since our results concern exclusively the whole line case, we need to address an eventual essential spectrum instability. As is the case in many applications, the essential spectrum can be easily computed by means of the Weyl's criterion. More specifically, in the conservative case, it turns out that the essential spectrum is constrained on the imaginary axes and as such is marginally stable. Thus, the real challenge is to study the eigenvalue problem, associated with (7).

The main objective of this work is to find a suitable criteria for the stability/instability of an abstract spectral problem in the form (7). We give a precise definition of spectral stability next.

Definition 1. *We say that the problem is spectrally unstable, if there exists* $u \in D(\mathcal{L}) = H^s(\mathbf{R}^1) \subset H^1(\mathbf{R}^1), u \neq 0$ *and* $\lambda : \Re\lambda > 0$*, so that* $\mathcal{L}u = \lambda u'$*. Otherwise, we say that the spectral problem* (7) *is stable.*

Next, we list a set of assumptions, which are necessary for our results. We require henceforth that $\Re\mathcal{L} = \mathcal{L}\Re$, that is $\mathcal{L}$ maps real valued into real valued functions, which will allow us to restrict our attention to real valued functions. Also,

$$\begin{cases} \mathcal{L} = \mathcal{L}^*, \sigma(\mathcal{L}) = \{-\sigma^2\} \cup \{0\} \cup \sigma_+(\mathcal{L}), \sigma_+(\mathcal{L}) \subset [\delta^2, \infty), \delta > 0 \\ \mathcal{L}f_0 = -\sigma^2 f_0, \ \dim[Ker(\mathcal{L} + \sigma^2)] = 1, \|f_0\| = 1 \\ \mathcal{L}\psi_0 = 0, \ \ \dim[Ker(\mathcal{L})] = 1, \|\psi_0\| = 1 \end{cases} \tag{8}$$

[2]We impose some additional technical assumptions later on.

Note that by our assumption, it follows that both f_0, ψ_0 are real valued.

Introduce the subspace $H_0 := span\{f_0, \psi_0\}^\perp$ and the action of the appropriate operators on it. Consider the spectral projection $P_{>0} : L^2 \to H_0 = \{f_0, \psi_0\}^\perp$, defined by

$$P_{>0}h = h - \langle h, f_0\rangle f_0 - \langle h, \psi_0\rangle \psi_0.$$

We will now establish the invertibility of the operator $P_{>0}(\mathcal{L} - \lambda\partial_x)P_{>0}$. In fact, we have the following *uniform in* $\lambda \in \mathbf{R}^1$ estimate for the inverse

Proposition 1. *Let λ be a real number. Then, the operator $P_{>0}(\mathcal{L} - \lambda\partial_x)P_{>0} : H_0 \to H_0$ is invertible and moreover, for every $g \in H_0$, we have*

$$\|(P_{>0}(\mathcal{L} - \lambda\partial_x)P_{>0})^{-1}g\|_{L^2} \leq \frac{1}{\delta^2}\|g\|_{L^2}. \tag{9}$$

where $\delta^2 = \inf \sigma_+(\mathcal{L})$.

Proof. The proof of the invertibility of $T = P_{>0}(\mathcal{L} - \lambda\partial_x)P_{>0}$ is classical. For a reference, one may look at Theorem 1 in [2], according to which, it suffices to check that the spectrum of the self-adjoint operator $\Re T = \frac{1}{2}(T + T^*)$ lies in the right hand plane $\Re z > 0$. But $\Re T = P_{>0}\mathcal{L}P_{>0} \geq \delta^2 > 0$ (here we use that λ is real), according to the assumption (8). Thus, T is invertible.

Regarding the estimate (9), take without loss of generality g to be real-valued and let $z = (\mathcal{L} - \lambda\partial_x)^{-1}g \in H_0$, also real-valued. We have that $(\mathcal{L} - \lambda\partial_x)z = g$. Taking dot product with z, yields

$$\|g\|\|z\| \geq \langle g, z\rangle = \langle(\mathcal{L} - \lambda\partial_x)z, z\rangle = \langle \mathcal{L}z, z\rangle \geq \delta^2\|z\|^2.$$

Thus, $\|z\| \leq \delta^{-2}\|g\|$, which is (9). □

Later on, we shall also need H^1 estimates of this inverse, but this does not appear to be a general fact[3], like Proposition 1. Thus, we shall need to assume it (see (11) below) in our general result and then check it for each particular example.

In view of the invertibility of $P_{>0}(\mathcal{L} - \lambda\partial_x)P_{>0}$, we will often write $(\mathcal{L} - \lambda\partial_x)^{-1}$ instead of $(P_{>0}(\mathcal{L} - \lambda\partial_x)P_{>0})^{-1}$. This will be particularly suitable for expressions of the form $\langle(\mathcal{L} - \lambda\partial_x)^{-1}f, g\rangle$, where $f, g \in H_0$.

1.1. **Instability results.**

In order to accomplish the required steps in the instability analysis, we make the following assumptions. We assume

$$\begin{cases} \psi_0 = g_0', g_0 \in L^2, \langle g_0, f_0\rangle \neq 0 \\ \psi_0 \in L^1(\mathbf{R}^1); f_0 \in L^1(\mathbf{R}^1) \cap H^{s-1}(\mathbf{R}^1), \\ \mathcal{L}g_0 \in L^1(\mathbf{R}^1) \cap H^{s-1}(\mathbf{R}^1). \end{cases} \tag{10}$$

We also assume that the operator $P_{>0}(\mathcal{L} - \lambda\partial_x)P_{>0}$ has H^1 bounds

$$\|P_{>0}(\mathcal{L} - \lambda\partial_x)^{-1}P_{>0}v\|_{H^1} \leq C(\lambda)\|v\|_{L^2}. \tag{11}$$

where λ is real and $C = C(\lambda)$ is a constant, which may grow as $\lambda \to \infty$, but is bounded on compact sets.

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[3]that is, something that follows from a generic assumption like (8)

Theorem 1. *Assume that $\mathcal{L}$ satisfies the assumptions* (8), (10), (11) *and*

$$\left\langle \mathcal{L}^{-1}\psi_0', \psi_0' \right\rangle > 0 \tag{12}$$

Then, the eigenvalue problem (7) *exhibits a real instability. That is, there exists $\lambda > 0$ and a real-valued $u \in D(\mathcal{L})$, so that $\mathcal{L}u = \lambda u'$.*

1.2. **Stability results.** In order to state our stability results (which are essentially complementary to the stability results), we shall need a somewhat different set of assumptions.

Recall the notion of index of a linear map, cf. Section 2.2, [13]. A linear map T has finite index, if $\dim(Ker(T)) < \infty$, $\operatorname{codim}(Ran(T)) < \infty$ and $ind(T) = \dim(Ker(T)) - \operatorname{codim}(Ran(T))$. In this terms, we require that for each complex $\lambda \notin i\mathbf{R}$,

$$ind(\mathcal{L} - \lambda\partial_x) = 0. \tag{13}$$

Moreover, we require that for each complex $\lambda \notin i\mathbf{R}$,

$$(\mathcal{L} - \lambda\partial_x)u = f \text{ has solution iff } \quad f \perp Ker(\mathcal{L} + \bar{\lambda}\partial_x). \tag{14}$$

We can restate (13), (14) in an even more concrete way. We are assuming, that we can find *orthonormal* systems $\{\psi_1, \ldots, \psi_n\}$ and $\{\psi_1^*, \ldots, \psi_n^*\}$ so that

$$Ker(\mathcal{L} - \lambda_0\partial_x) = span\{\psi_1, \ldots, \psi_n\}, Ker(\mathcal{L} + \bar{\lambda}_0\partial_x) = span\{\psi_1^*, \ldots, \psi_n^*\}.$$

so that the equation $(\mathcal{L} - \lambda_0\partial_x)u = f$ has a solution, if and only if $\left\langle f, \psi_j^* \right\rangle = 0, j = 1, \ldots, n$. Moreover, it is clear that one can then find a solution u, which belongs to $span\{\psi_1, \ldots, \psi_n\}^\perp$. We need to however also assume that this solution is unique. That is

$$\forall f \in span\{\psi_1^*, \ldots, \psi_n^*\}^\perp, \exists! g \in span\{\psi_1, \ldots, \psi_n\}^\perp : (\mathcal{L} - \lambda\partial_x)g = f. \tag{15}$$

The assumptions (13), (14) and (15) are of course the statement of the Fredholm alternative for operators in the form $I + \mathcal{K}$, where $\mathcal{K}$ is compact operator. Note however that the operators involved here, namely $\mathcal{L} \pm \lambda\partial_x$ are unbounded. In the applications, these assumptions are checked by suitably rewriting such equations in the form $I + \mathcal{K}$ for a suitable compact operator, involving resolvents of constant coefficient operators. Finally, the condition (15) can be expressed in terms of the boundedness of the operator $(\mathcal{L} - \lambda\partial_x)^{-1} : span\{\psi_1^*, \ldots, \psi_n^*\}^\perp \to span\{\psi_1, \ldots, \psi_n\}^\perp$.

In order to be able to control the essential spectrum of the operators under consideration, we need the following assumption: for some $s > 1$,

$$\left\{ \begin{array}{l} \mathcal{L} = \mathcal{L}_0 + \mathcal{K}, \widehat{\mathcal{L}_0 f}(\xi) = q_0(\xi)\hat{f}(\xi), \\ q_0(\xi) \geq \delta^2, \lim_{|\xi|\to\infty} \frac{q_0(\xi)}{|\xi|^s} = c_0 > 0. \end{array} \right. \tag{16}$$

and $\mathcal{K}(-\partial_x^2 + 1)^{1/4}$ is a relatively compact perturbation of $\mathcal{L}_0$. That is, $\mathcal{K}(-\partial_x^2 + 1)^{1/4}\mathcal{L}_0^{-1}$ is compact. This assumption, while slightly stronger than the usual assumptions (like $\mathcal{K}$ is a relatively compact perturbation of $\mathcal{L}_0$) is nevertheless mild enough to cover virtually all cases of interest, especially since we have already assumed that $D(\mathcal{L}) = H^s, s > 1$.

Theorem 2. *Assume that $\mathcal{L}$ satisfies the assumptions* (8), (13), (14), (15), (16) *and*

$$\left\langle \mathcal{L}^{-1}\psi_0', \psi_0' \right\rangle < 0. \tag{17}$$

Then, the eigenvalue problem (7) *is spectrally stable. That is, no solution u to* (7) *exists.*

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2. Proof of the Instability criteria

In this section, we establish Theorem 1. We closely follow the method of our previous articles [24], [25], where we have developed a similar index theory for quadratic pencils. The idea of the method is to construct an appropriate determinant type function $G(\lambda)$, which we consider only over the reals, so that $G(\lambda) = 0$ if and only if λ is an eigenvalue for (7). We show that this is the case, only if the condition (12) is satisfied. Unfortunately, following this argument, we cannot conclude that the stability occurs when a condition opposite to (12) holds. Even though this turns out to be the case, we are forced to provide a completely new line of reasoning in Section 3 below. The reason is that there is no general theory available for eigenvalue problems in the form (7), which guarantees that the unstable eigenvalue must be necessarily real. So, while we can rule out real instabilities in the case of (17), we cannot rule out complex instabilities, which necessitates the arguments of Section 3.

2.1. **Introduction of the Evans like function G.** Since we take the spectral parameter to be real and since $\mathcal{L}$ maps real-valued functions into themselves, it follows that if (7) has solutions, then one can produce λ real and u real, so that $\mathcal{L}u = \lambda u'$. Thus, we seek for eigenfunctions in the form $u = a_0 f_0 + b_0 \psi_0 + v$, where f_0, ψ_0 are the eigenfunctions of $\mathcal{L}$, corresponding to the negative and zero eigenvalue, a_0, b_0 are reals and $v \in \{f_0, \psi_0\}^\perp =: H_0$.

We obtain the following

$$-a_0 \sigma^2 f_0 + \mathcal{L}v = \lambda(a_0 f_0' + b_0 \psi_0' + v'). \tag{18}$$

Taking dot product with f_0 yields (note $\langle f_0', f_0\rangle = 0$, $\langle \mathcal{L}v, f_0\rangle = \langle v, \mathcal{L}f_0\rangle = -\sigma^2 \langle v, f_0\rangle = 0$) $-a_0\sigma^2 - \lambda b_0 \langle \psi_0', f_0\rangle - \lambda \langle v', f_0\rangle = 0$, which is the same (after integration by parts) as

$$-a_0\sigma^2 - \lambda b_0 \langle \psi_0', f_0\rangle + \lambda \langle v, f_0'\rangle = 0. \tag{19}$$

Taking dot product in (18) with ψ_0 results in $-a_0\lambda \langle f_0', \psi_0\rangle - \lambda \langle v', \psi_0\rangle = 0$, whence since $\lambda \neq 0$,

$$-a_0 \langle f_0', \psi_0\rangle + \langle v, \psi_0'\rangle = 0. \tag{20}$$

The remaining relations in (18) are equivalent to taking $P_{>0}$ in both sides of it. We obtain

$$P_{>0}(\mathcal{L} - \lambda\partial_x)v = \lambda P_{>0}(a_0 f_0' + b_0\psi_0'). \tag{21}$$

Note that if the operator $P_{>0}(\mathcal{L} - \lambda\partial_x)P_{>0} : H_0 \to H_0$ is invertible, (21) may be resolved

$$v = \lambda(\mathcal{L} - \lambda\partial_x)^{-1}[P_{>0}(a_0 f_0' + b_0\psi_0')] \tag{22}$$

Having now (22), we may use it back in (19) and (20). We obtain the pair of equations

$$a_0(\lambda^2 \left\langle (\mathcal{L} - \lambda\partial_x)^{-1} P_{>0} f_0', f_0'\right\rangle - \sigma^2) + b_0(\lambda^2 \left\langle (\mathcal{L} - \lambda\partial_x)^{-1} P_{>0}\psi_0', f_0'\right\rangle - \lambda \langle \psi_0', f_0\rangle) = 0$$
$$a_0(\lambda \left\langle (\mathcal{L} - \lambda\partial_x)^{-1} P_{>0} f_0', \psi_0'\right\rangle - \langle f_0', \psi_0\rangle) + b_0\lambda \left\langle (\mathcal{L} - \lambda\partial_x)^{-1} P_{>0}\psi_0', \psi_0'\right\rangle = 0$$

This homogeneous system is a compatibility condition for a_0, b_0. In other words, a (a non-trivial) solution exists if the determinant is non-zero. This gives us exactly our Evans-like function G. In order to have concise notations, let

$$a_{11} := \left\langle (\mathcal{L}-\lambda\partial_x)^{-1} P_{>0} f_0', f_0' \right\rangle, \quad a_{12} := \left\langle (\mathcal{L}-\lambda\partial_x)^{-1} P_{>0} \psi_0', f_0' \right\rangle$$
$$a_{21} := \left\langle (\mathcal{L}-\lambda\partial_x)^{-1} P_{>0} f_0', \psi_0' \right\rangle, \quad a_{22} := \left\langle (\mathcal{L}-\lambda\partial_x)^{-1} P_{>0} \psi_0', \psi_0' \right\rangle$$

Note $a_{11} := \langle (\mathcal{L}-\lambda\partial_x)^{-1} P_{>0} f_0', f_0' \rangle = \langle (\mathcal{L}-\lambda\partial_x)^{-1} P_{>0} f_0', P_{>0} f_0' \rangle$ and similar for the others.

Consider

$$\begin{vmatrix} \lambda^2 \langle (\mathcal{L}-\lambda\partial_x)^{-1} P_{>0} f_0', f_0' \rangle - \sigma^2 & \lambda^2 \langle (\mathcal{L}-\lambda\partial_x)^{-1} P_{>0} \psi_0', f_0' \rangle - \lambda \langle \psi_0', f_0 \rangle \\ \lambda \langle (\mathcal{L}-\lambda\partial_x)^{-1} P_{>0} f_0', \psi_0' \rangle - \langle f_0', \psi_0 \rangle & \lambda \langle (\mathcal{L}-\lambda\partial_x)^{-1} P_{>0} \psi_0', \psi_0' \rangle \end{vmatrix} =$$
$$= \lambda(\lambda^2(a_{11}a_{22} - a_{12}a_{21}) + \lambda \langle f_0', \psi_0 \rangle (a_{12} - a_{21}) + \langle f_0', \psi_0 \rangle^2 - \sigma^2 a_{22}).$$

Thus, we have proved the following

Proposition 2. *Under the assumptions made in Theorem 7, the eigenvalue problem* (7) *has a solution $\lambda > 0$ and $u \in D(\mathcal{L})$ (i.e. instability), if the function*

$$G(\lambda) := \lambda^2(a_{11}a_{22} - a_{12}a_{21}) + \lambda \langle f_0', \psi_0 \rangle (a_{12} - a_{21}) + \langle f_0', \psi_0 \rangle^2 - \sigma^2 a_{22}$$

vanishes somewhere in $(0, \infty)$*.*

In order to check that G vanish, we will show that it is continuous and then, that it changes its sign in $(0, \infty)$. We start with the continuity.

Proposition 3. *The function $G : (0, \infty) \to \mathbf{R}^1$ defined in Propostion 2 is a continuous function.*

Proof. By the form of G, it suffices to check that each of $a_{ij}(\lambda), i, j = 1, 2$ are continuous functions of λ. Let f_1, f_2 be arbitrary smooth functions in H_0. We show that $\lambda \to \langle (\mathcal{L}-\lambda\partial_x)^{-1} f_1, f_2 \rangle := m(\lambda)$ is continuous, which implies the Proposition. Indeed, taking the difference of the functional values, we have by the resolvent identity

$$\begin{aligned} m(\lambda_1) - m(\lambda_2) &= \left\langle (\mathcal{L}-\lambda_1\partial_x)^{-1} f_1, f_2 \right\rangle - \left\langle (\mathcal{L}-\lambda_2\partial_x)^{-1} f_1, f_2 \right\rangle = \\ &= (\lambda_1 - \lambda_2) \left\langle (\mathcal{L}-\lambda_1\partial_x)^{-1} P_{>0} \partial_x P_{>0} (\mathcal{L}-\lambda_2\partial_x)^{-1} f_1, f_2 \right\rangle. \end{aligned}$$

Thus, we may estimate, by Proposition 1

$$|m(\lambda_1) - m(\lambda_2)| \leq C|\lambda_1 - \lambda_2| \|f_2\|_{L^2} \|\partial_x P_{>0} (\mathcal{L}-\lambda_2\partial_x)^{-1} f_1\|_{L^2}.$$

By assumption 11, we can now conclude that for each $\lambda_1 > 0$, $\lim_{\lambda \to \lambda_1} m(\lambda) = m(\lambda_1)$. □

Next, we need to show that G changes signs. To that end, we first consider its behavior at ∞.

2.2. The behavior of $G(\lambda)$ as $\lambda \to \infty$.

We start with a lemma about the long term behavior $a_{12}(\lambda), a_{21}(\lambda)$.

Lemma 1.

$$\lim_{\lambda \to \infty} a_{12}(\lambda) = 0 = \lim_{\lambda \to \infty} a_{21}(\lambda).$$

Remark: This result is only a preliminary step. We have in fact a more precise estimate for the large λ behavior of a_{12}, a_{21}, see Lemma 4 below.

Proof. Let $v \in H_0$ be defined via

$$(\mathcal{L} - \lambda P_{>0}\partial_x)v = P_{>0}\psi_0'. \tag{23}$$

We need to show that $a_{12}(\lambda) = \langle v, f_0' \rangle$ converges to 0, as $\lambda \to \infty$. Take a dot product with with[4]

$$P_{>0}g_0 = g_0 - \langle g_0, f_0 \rangle f_0.$$

We have

$$\langle \mathcal{L}v, P_{>0}g_0 \rangle - \lambda \langle v', g_0 - \langle g_0, f_0 \rangle f_0 \rangle = \langle P_{>0}\psi_0', g_0 \rangle . \tag{24}$$

But,

$$| \langle \mathcal{L}v, P_{>0}g_0 \rangle | = | \langle v, \mathcal{L}P_{>0}g_0 \rangle | \leq \|v\|_{L^2} \|\mathcal{L}P_{>0}g_0\|_{L^2} \leq C\|v\|_{L^2}\|g_0\|_{H^s}(1 + \|f_0\|_{L^2}^2).$$

From Proposition 1, we have that $\|v\|_{L^2} \leq C_\delta \|P_{>0}\psi_0'\|_{L^2}$. Thus, the contribution of $\langle \mathcal{L}v, P_{>0}g_0 \rangle$ is uniformly bounded in λ. Same is true for the right hand side, since

$$| \langle P_{>0}\psi_0', g_0 \rangle | \leq \|P_{>0}\psi_0'\|_{L^2}\|g_0\|_{L^2}.$$

Next, $\langle v', g_0 \rangle = - \langle v, g_0' \rangle = - \langle v, \psi_0 \rangle = 0$, since $v \in H_0 = \{f_0, \psi_0\}^\perp$. Finally, $\langle g_0, f_0 \rangle \neq 0$ by assumption, while $\langle v', f_0 \rangle = - \langle v, f_0' \rangle = -a_{12}(\lambda)$. Thus, the equation (24) takes the form

$$\lambda a_{12}(\lambda) \langle g_0, f_0 \rangle = O(1),$$

whence $\lim_{\lambda\to\infty} a_{12}(\lambda) = 0$. The statement for $a_{21}(\lambda)$ follows in a similar way, since one can write

$$a_{21}(\lambda) = \left\langle (\mathcal{L} - \lambda P_{>0}\partial_x)^{-1} P_{>0} f_0', P_{>0}\psi_0' \right\rangle = \left\langle f_0', (\mathcal{L} + \lambda P_{>0}\partial_x)^{-1} P_{>0}\psi_0' \right\rangle$$

and run the same argument (with λ replaced by $-\lambda$ in (23)). □

Our next lemma concerns the behavior of $a_{22}(\lambda)$ for $\lambda >> 1$. .

Lemma 2. *Under the assumption* (10), $\lim_{\lambda\to\infty} a_{22}(\lambda) = 0$

Proof. Starting with the equation

$$(\mathcal{L} - \lambda P_{>0}\partial_x P_{>0})v = P_{>0}\psi_0' \tag{25}$$

where $v \in H_0$, we need to show that $a_{22} = \langle v, \psi_0' \rangle \to 0$, as $\lambda \to \infty$. Note first that by (9),

$$\|v\|_{L^2} = \|(\mathcal{L} - \lambda P_{>0}\partial_x P_{>0})^{-1} P_{>0}\psi_0'\|_{L^2} \leq C\delta^{-2}\|\psi_0'\|_{L^2}.$$

For the next step, we would ideally work with the antiderivative of f_0. Unfortunately, in the applications, f_0 is in general a positive function and does not have localized antiderivative. Instead, we introduce the following bounded function,

$$h^N(x) = \chi(x/N) \int_0^x f_0(y)dy,$$

[4]Note that the term $\langle g_0, \psi_0 \rangle \psi_0$ is missing, since $\langle g_0, \psi_0 \rangle = \langle g_0, g_0' \rangle = 0$.

where $N >> 1$ and χ is an even $C_0^\infty(\mathbf{R}^1)$ function, decreasing in $(0,\infty)$, so that $\chi(x) = 1, -1 < x < 1$ and $\chi(x) = 0, |x| > 2$. Taking a dot product of (25) with h^N, we have

$$\left\langle \mathcal{L}v, h^N \right\rangle - \lambda \left\langle v', P_{>0}h^N \right\rangle = \left\langle P_{>0}\psi_0', h^N \right\rangle. \tag{26}$$

We now estimate various terms in (26). We have

$$|\left\langle \mathcal{L}v, h^N \right\rangle| = |\left\langle v, \mathcal{L}h^N \right\rangle| \leq \|v\|_{L^2}\|\mathcal{L}h^N\|_{L^2} \leq C\delta^{-2}\|\psi_0'\|_{L^2}\|h^N\|_{H^s} \leq C_N,$$

where in the last step, we have used that $\|h^N\|_{H^s} \leq C(\sqrt{N}\|f_0\|_{L^1} + \|f_0\|_{H^{s-1}})$. Note that this bound is independent on λ. Next,

$$|\left\langle P_{>0}\psi_0', h^N \right\rangle| \leq C\|P_{>0}\psi_0'\|_{L^2}\|h^N\|_{L^2} \leq C_N.$$

For the remaining term, we have

$$\left\langle v', P_{>0}h^N \right\rangle = \left\langle v', h^N \right\rangle - \langle v', f_0\rangle \left\langle f_0, h^N \right\rangle - \langle v', \psi_0\rangle \left\langle \psi_0, h^N \right\rangle$$

We now estimate various terms that arise.

$$\left\langle v', h^N \right\rangle = -\left\langle v, \partial_x h^N \right\rangle = -\frac{1}{N}\int v(x)\chi'(x/N)(\int_0^x f_0(y)dy)dx - \int v(x)f_0(x)\chi(x/N)dx$$

Recalling that $\int v(x)f_0(x)dx = \langle v, f_0\rangle = 0$, we have

$$|\int v(x)f_0(x)\chi(x/N)dx| = |\int v(x)f_0(x)(1-\chi(x/N))dx| \leq \|v\|_{L^2}\|f_0\|_{L^2(|x|>N)},$$

which converges to zero as $N \to \infty$. Also,

$$|\frac{1}{N}\int v(x)\chi'(x/N)(\int_0^x f_0(y)dy)dx| \leq CN^{-1/2}\|f_0\|_{L^1}\|v\|_{L^2} \leq C\delta^{-2}N^{-1/2}\|f_0\|_{L^1}\|\psi_0'\|_{L^2}.$$

Here again, by Proposition 1, we have used that $\|v\|_{L^2} \leq C\delta^{-2}\|P_{>0}\psi_0'\|_{L^2}$. Hence, $|\left\langle v', h^N \right\rangle| = o(1/N)$, with constants independent of λ.

Consider now $\left\langle f_0, h^N \right\rangle$. We have

$$|\left\langle f_0, h^N \right\rangle| \leq \|f_0\|_{L^1}\|h^N\|_{L^\infty} \leq \|f_0\|_{L^1}^2.$$

Also

$$\langle v', f_0\rangle = -\langle v, f_0'\rangle = -\langle v, P_{>0}f_0'\rangle = -\left\langle (\mathcal{L} - P_{>0}\partial_x P_{>0})^{-1}[P_{>0}\psi_0'], P_{>0}f_0' \right\rangle = -a_{12}(\lambda).$$

By Lemma 1, we conclude that $\left\langle f_0, h^N \right\rangle \langle v', f_0\rangle = o(1/\lambda)$.

Next, by (10), we have

$$\begin{aligned} \left\langle \psi_0, h^N \right\rangle &= \left\langle g_0', h^N \right\rangle = -\left\langle g_0, \partial_x h^N \right\rangle = -\frac{1}{N}\int g_0(x)\chi'(x/N)(\int_0^x f_0(y)dy)dx \\ &- \int g_0(x)\chi(x/N)f_0(x)dx \end{aligned}$$

The first term is again $o(1/N)$, because it can be estimated by $CN^{-1/2}\|g_0\|_{L^2}\|f_0\|_{L^1}$. For the second term, note that by the assumption in (10), $\int g_0 f_0 = \langle g_0, f_0\rangle \neq 0$, whence

$$\int g_0(x)\chi(x/N)f_0(x)dx = \langle f_0, g_0\rangle - \int g_0(x)f_0(x)(1-\chi(x/N))dx.$$

Note $|\int g_0(x) f_0(x)(1-\chi(x/N))dx| \leq C\|g_0\|_{L^2}\|f_0\|_{L^2(|x|>N)} = o(1/N)$. Putting all this information in (26) yields

$$|\langle v, \psi_0' \rangle \langle g_0, f_0 \rangle + o(1/N) + o(1/\lambda)| \leq \frac{C_N}{\lambda}.$$

It follows that $\lim_{\lambda\to\infty} a_{22}(\lambda) = \lim_{\lambda\to\infty} \langle v(\lambda), \psi_0' \rangle = o(1/N)$ for all large N and hence $\lim_{\lambda\to\infty} a_{22}(\lambda) = 0$. □

We also need the following result.

Lemma 3. *Let $m \in L^1 \cap H^{s-1}$. Then,*

$$\lim_{\lambda\to\infty} \left\langle (\mathcal{L} - \lambda P_{>0}\partial_x P_{>0})^{-1} P_{>0}\psi_0', P_{>0} m \right\rangle = 0.$$

Proof. The proof of Lemma 3 proceeds similarly to Lemma 2. For conciseness, let

$$a(\lambda) := \left\langle (\mathcal{L} - \lambda P_{>0}\partial_x P_{>0})^{-1} P_{>0}\psi_0', P_{>0} m \right\rangle.$$

We start with the same $v \in H_0$, as defined in (25). Note that $a(\lambda) = \langle v, m \rangle$. For large N, consider the function

$$q^N(x) = \chi(x/N) \int_0^x m(y) dy.$$

This is clearly a bounded compactly supported function, since $m \in L^1$. Take a dot product of q^N with both sides[5] of (25). We have

$$\left\langle \mathcal{L}v, q^N \right\rangle - \lambda \left\langle v', P_{>0} q^N \right\rangle = \left\langle P_{>0}\psi_0', q^N \right\rangle. \tag{27}$$

We have

$$|\left\langle \mathcal{L}v, q^N \right\rangle| = |\left\langle v, \mathcal{L}q^N \right\rangle| \leq \|v\|_{L^2}\|\mathcal{L}q^N\|_{L^2} \leq C\delta^{-2}\|v\|_{L^2}\|q^N\|_{H^s}.$$

By Proposition 1, we have $\|v\|_{L^2} \leq C\|P_{>0}\psi_0'\|_{L^2}$. In addition, $\|q^N\|_{H^s} \leq C(\sqrt{N}\|m\|_{L^1} + \|m\|_{H^{s-1}})$. Thus,

$$|\left\langle \mathcal{L}v, q^N \right\rangle| \leq C_N.$$

Similarly,

$$|\left\langle P_{>0}\psi_0', q^N \right\rangle| \leq \|P_{>0}\psi_0'\|_{L^2}\|q^N\|_{L^2} \leq C_N.$$

Next, we have

$$\begin{aligned} \left\langle v', P_{>0} q^N \right\rangle &= \left\langle v', q^N - \left\langle q^N, f_0 \right\rangle f_0 - \left\langle q^N, \psi_0 \right\rangle \psi_0 \right\rangle = \\ &= \left\langle v', q^N \right\rangle - \left\langle q^N, f_0 \right\rangle \langle v', f_0 \rangle - \left\langle q^N, \psi_0 \right\rangle \langle v', \psi_0 \rangle. \end{aligned}$$

Now,

$$|\left\langle q^N, f_0 \right\rangle| + |\left\langle q^N, \psi_0 \right\rangle| \leq \|q^N\|_{L^\infty}(\|f_0\|_{L^1} + \|\psi_0\|_{L^1}) \leq \|m\|_{L^1}(\|f_0\|_{L^1} + \|\psi_0\|_{L^1})$$

On the other hand, by Lemma 1, Lemma 2,

$$\begin{aligned} \langle v', f_0 \rangle &= -\langle v, f_0' \rangle = -\langle v, P_{>0} f_0' \rangle = -a_{12}(\lambda) = o(1/\lambda) \\ \langle v', \psi_0 \rangle &= -\langle v, \psi_0' \rangle = -\langle v, P_{>0}\psi_0' \rangle = -a_{22}(\lambda) = o(1/\lambda). \end{aligned}$$

[5]Since the equation (25) is projected over H_0 anyway, we may choose whether to enter $P_{>0}$ in front of q^N or not

Finally,

$$\left\langle v', q^N \right\rangle = -\left\langle v, \partial_x q^N \right\rangle = -\int v(x)\chi(x/N)m(x)dx - N^{-1}\int v(x)\chi'(x/N)\int_0^x m(y)dy.$$

But now by Cauchy-Schwartz

$$|N^{-1}\int v(x)\chi'(x/N)\int_0^x m(y)dy| \leq CN^{-1/2}\|m\|_{L^1}\|v\|_{L^2} = o(1/N)$$

and

$$\int v(x)\chi(x/N)m(x)dx = \langle v, m\rangle - \int v(x)(1-\chi(x/N))m(x)dx,$$

for which we have that

$$|\int v(x)(1-\chi(x/N))m(x)dx| \leq \|v\|_{L^2}\|m\|_{L^2(|x|>N)} = o(1/N).$$

Thus, $\left\langle v', q^N \right\rangle = -\langle v, m\rangle + o(1/N) + o(1/\lambda) = -a(\lambda) + o(1/N) + o(1/\lambda)$. Thus, (27) implies

$$|a(\lambda) + o(1/N) + o(1/\lambda)| \leq \frac{C_N}{\lambda}.$$

Taking a limit in λ implies $\lim_{\lambda\to\infty} a(\lambda) = 0$. □

Our next lemma gives a more precise behavior of the functions a_{12}, a_{21} for large λ.

Lemma 4. *Assuming that* $\mathcal{L}g_0 \in L^1 \cap H^{s-1}$,

$$\lim_{\lambda\to\infty} \lambda a_{12}(\lambda) = \frac{1 - \langle g_0, f_0\rangle \langle f_0', \psi_0\rangle}{\langle g_0, f_0\rangle} = -\lim_{\lambda\to\infty} \lambda a_{21}(\lambda).$$

Proof. As in the proof of Lemma 1, we consider (23). Take a dot product with $P_{>0}g_0$, so that we have (24). We now proceed to analyze the different terms in (24). We have by Lemma 3, with $m = \mathcal{L}g_0$,

$$\langle \mathcal{L}v, P_{>0}g_0\rangle = \langle v, \mathcal{L}P_{>0}g_0\rangle | = \left\langle (\mathcal{L} - \lambda P_{>0}\partial_x P_{>0})^{-1}P_{>0}\psi_0', P_{>0}\mathcal{L}g_0\right\rangle \to 0.$$

Next, since $\langle v, g_0'\rangle = \langle v, \psi_0\rangle = 0$, we have

$$\langle v', g_0 - \langle g_0, f_0\rangle f_0\rangle = -\langle v, g_0'\rangle + \langle g_0, f_0\rangle \langle v, f_0'\rangle = \langle g_0, f_0\rangle \langle v, f_0'\rangle.$$

For the right hand-side of (24),

$$\begin{aligned} \langle P_{>0}\psi_0', g_0\rangle &= \langle \psi_0', g_0 - \langle g_0, f_0\rangle f_0\rangle = -\langle \psi_0, g_0'\rangle + \langle g_0, f_0\rangle \langle f_0', \psi_0\rangle = \\ &= -\|\psi_0\|^2 + \langle g_0, f_0\rangle \langle f_0', \psi_0\rangle. \end{aligned}$$

Putting all this information back in (24) yields

$$\lambda a_{12}(\lambda) \langle g_0, f_0\rangle + o(1/\lambda) = 1 - \langle g_0, f_0\rangle \langle f_0', \psi_0\rangle.$$

Taking limits in $\lambda \to \infty$, yields the relation

$$\lim_{\lambda\to\infty} \lambda a_{12}(\lambda) = \frac{1 - \langle g_0, f_0\rangle \langle f_0', \psi_0\rangle}{\langle g_0, f_0\rangle}.$$

Repeating the arguments above (note the only difference is the plus sign in front of the important term $P_{>0}\partial_x P_{>0}$), we achieve

$$\lim_{\lambda\to\infty} \lambda a_{21}(\lambda) = -\frac{1 - \langle g_0, f_0\rangle \langle f_0', \psi_0\rangle}{\langle g_0, f_0\rangle}$$

whence Lemma 4 is proved in full. For future reference, note that

$$\lim_{\lambda\to\infty} \frac{\lambda}{2}(a_{12} - a_{21}) = \frac{1}{\langle g_0, f_0\rangle} - \langle f_0', \psi_0\rangle . \tag{28}$$

□

We continue with the study of $G(\lambda)$ for large values of λ. Observe that for each real valued $z \in H_0$, we have

$$\left\langle (\mathcal{L} - \lambda\partial_x)^{-1} z, z\right\rangle \geq 0. \tag{29}$$

Indeed, take $z = P_{>0}(\mathcal{L} - \lambda\partial_x)f$, f real-valued and $f \in H_0$. It will suffice to show that $\langle (\mathcal{L} - \lambda\partial_x)f, f\rangle > 0$. But

$$\langle (\mathcal{L} - \lambda\partial_x)f, f\rangle = \langle \mathcal{L}f, f\rangle - \lambda \langle f', f\rangle = \langle \mathcal{L}f, f\rangle \geq \delta^2\|f\|^2 \geq 0,$$

whence (29) follows. Now, denoting $T = (P_{>0}(\mathcal{L} - \lambda\partial_x)P_{>0})^{-1}$, we have that for each μ real and $f_1, f_2 \in H_0$ real-valued,

$$\langle T(\mu f_1 + f_2), (\mu f_1 + f_2)\rangle \geq 0$$

Thus,

$$\mu^2 \langle Tf_1, f_1\rangle + \mu(\langle Tf_1, f_2\rangle + \langle Tf_2, f_1\rangle) + \langle Tf_2, f_2\rangle \geq 0.$$

This is a quadratic function in μ, which is non-negative for all values of μ. Hence, its determinant must be non-positive

$$(\langle Tf_1, f_2\rangle + \langle Tf_2, f_1\rangle)^2 \leq 4 \langle Tf_1, f_1\rangle \langle Tf_2, f_2\rangle$$

Adding and subtracting $4 \langle Tf_1, f_2\rangle \langle Tf_2, f_1\rangle$ and some algebra leads to

$$\langle Tf_1, f_1\rangle \langle Tf_2, f_2\rangle - \langle Tf_1, f_2\rangle \langle Tf_2, f_1\rangle \geq \frac{1}{4}(\langle Tf_1, f_2\rangle - \langle Tf_2, f_1\rangle)^2 \tag{30}$$

Applying this last inequality to $f_1 = P_{>0}f_0'$ and $f_2 = P_{>0}\psi_0'$ yields

$$a_{11}a_{22} - a_{12}a_{21} \geq \frac{1}{4}(a_{12} - a_{21})^2.$$

Inserting this inequality in the definition of G results in the following inequality

$$G(\lambda) \geq \left(\frac{\lambda}{2}(a_{12} - a_{21}) + \langle f_0', \psi_0\rangle\right)^2 - \sigma^2 a_{22}.$$

According to Lemma 2 and Lemma 4 (more specifically (28)), we have

$$\limsup_{\lambda\to\infty} G(\lambda) \geq \lim_{\lambda\to\infty} \left(\frac{\lambda}{2}(a_{12} - a_{21}) + \langle f_0', \psi_0\rangle\right)^2 - \sigma^2 \lim_{\lambda\to\infty} a_{22}(\lambda) = \frac{1}{\langle g_0, f_0\rangle^2} > 0.$$

Thus $G(\lambda)$ achieves a positive value somewhere on $(0, \infty)$.

2.3. **Conclusion of the proof of the instability criterion.** We now look at the limit $\lim_{\lambda\to 0+} G(\lambda)$. If we show that $\lim_{\lambda\to 0+} G(\lambda) < 0$, this would imply that the continuous function G changes sign in $(0,\infty)$ and hence $G(\lambda_0)=0$ for some $\lambda_0>0$. This of course implies the instability and the proof of Theorem 1 would be complete.

To that end, note first that the operator $P_{>0}\partial_x P_{>0}\mathcal{L}^{-1}P_{>0} : H_0 \to H_0$ is well-defined and bounded. Hence, we may write

$$\begin{aligned}(P_{>0}(\mathcal{L}-\lambda\partial_x)P_{>0})^{-1} &= [(Id-\lambda P_{>0}\partial_x P_{>0}\mathcal{L}^{-1}P_{>0})P_{>0}\mathcal{L}P_{>0}]^{-1} = \\ &= P_{>0}\mathcal{L}^{-1}P_{>0}\sum_{k=0}^{\infty}\lambda^k (P_{>0}\partial_x P_{>0}\mathcal{L}^{-1}P_{>0})^k,\end{aligned}$$

for all values of $\lambda : |\lambda| < \|P_{>0}\partial_x P_{>0}\mathcal{L}^{-1}P_{>0}\|_{B(H_0)}^{-1}$, in particular for all small values of λ. Thus,

$$\lim_{\lambda\to 0+}\|(P_{>0}(\mathcal{L}-\lambda\partial_x)P_{>0})^{-1} - P_{>0}\mathcal{L}^{-1}P_{>0}\|_{B(H_0)} = 0.$$

Thus, it becomes very easy to compute the limit, $\lim_{\lambda\to 0} G(\lambda)$. Indeed,

$$\begin{aligned}&\lim_{\lambda\to 0} a_{11}(\lambda) = \left\langle \mathcal{L}^{-1}P_{>0}f_0', P_{>0}f_0'\right\rangle, \quad \lim_{\lambda\to 0} a_{12}(\lambda) = \left\langle \mathcal{L}^{-1}P_{>0}\psi_0', P_{>0}f_0'\right\rangle \\ &\lim_{\lambda\to 0} a_{21}(\lambda) = \left\langle \mathcal{L}^{-1}P_{>0}f_0', P_{>0}\psi_0'\right\rangle, \quad \lim_{\lambda\to 0} a_{22}(\lambda) = \left\langle \mathcal{L}^{-1}P_{>0}\psi_0', P_{>0}\psi_0'\right\rangle\end{aligned}$$

Thus,

$$\lim_{\lambda\to 0+} G(\lambda) = \langle f_0', \psi_0\rangle^2 - \sigma^2\left\langle \mathcal{L}^{-1}P_{>0}\psi_0', P_{>0}\psi_0'\right\rangle \tag{31}$$

But how do we compute $P_{>0}\mathcal{L}^{-1}P_{>0}\psi_0'$? Note that $\mathcal{L}^{-1}[P_{>0}\psi_0'] \in L^2$ is well-defined (but it does not necessarily belong to H_0!), since $\psi_0' \perp \psi_0$. We claim that

$$P_{>0}\mathcal{L}^{-1}P_{>0}\psi_0' = \mathcal{L}^{-1}[\psi_0'] - \left\langle \mathcal{L}^{-1}[\psi_0'], f_0\right\rangle f_0 =: Z$$

Indeed, the right hand side Z of the formula belongs to H_0, as it should according to the definition of $P_{>0}\mathcal{L}^{-1}P_{>0}$. Next, we need to check that it satisfies $\mathcal{L}Z = P_{>0}\psi_0'$. Indeed,

$$\begin{aligned}\mathcal{L}[Z] &= \mathcal{L}[\mathcal{L}^{-1}[\psi_0'] - \left\langle \mathcal{L}^{-1}[\psi_0'], f_0\right\rangle f_0] = \psi_0' + \sigma^2\left\langle \mathcal{L}^{-1}[\psi_0'], f_0\right\rangle f_0 = \\ &= \psi_0' + \sigma^2\left\langle \psi_0', \mathcal{L}^{-1}f_0\right\rangle f_0 = \psi_0' - \langle \psi_0', f_0\rangle f_0 = P_{>0}\psi_0'.\end{aligned}$$

Thus,

$$\left\langle \mathcal{L}^{-1}P_{>0}\psi_0', P_{>0}\psi_0'\right\rangle = \left\langle \mathcal{L}^{-1}\psi_0', \psi_0'\right\rangle + \frac{1}{\sigma^2}\langle f_0', \psi_0\rangle^2.$$

Plugging this inside (31), we get

$$\lim_{\lambda\to 0+} G(\lambda) = -\sigma^2\left\langle \mathcal{L}^{-1}\psi_0', \psi_0'\right\rangle < 0,$$

since $\langle \mathcal{L}^{-1}\psi_0', \psi_0'\rangle > 0$ per (12). The proof of Theorem 1 is now complete.

3. Stability criteria: Proof of Theorem 2

Before we move on to the specifics of the proof, let us explain the idea behind it. First, the form of the eigenvalue problem (7) is in the form $\mathcal{L}u = \lambda \mathcal{J}u$, where $\mathcal{J} = \partial_x$ is skew-adjoint, while $\mathcal{L}^* = \mathcal{L}$. This is somewhat reminiscent of the Kapitula-Kevrekidis-Sandstede setup (KKS for short), [11], [12] for their index counting formula, which we review below in Section 3.1. Related work has been done by Chugunova-Pelinovsky, [4], where the generalized eigenvalue problem is considered in the context of Krein spaces.

For various reasons, the aforementioned index theories do not apply here, the main reason being that $\mathcal{J} = \partial_x$ is not an invertible operator. In a more recent paper [14], similar issues were investigated, for KdV type eigenvalue problems in the form $\partial_x \mathcal{L}u = \lambda u$, which are also not covered by the KKS theory.

For the stability criteria, we argue by contradiction. More specifically, assuming the condition (17) and assuming instability, we construct (via a Lyapunov-Schmidt reduction argument) a family of approximate eigenvalue problems, which also support unstable modes. However, these approximate eigenvalue problems are within the range of the KKS theory and hence its instability prediction holds true. As a limit, such an inequality turns out to contradict (17) and this leads us to the proof of the stability in this case.

3.1. Some preliminaries.

We briefly review the main result in [11], [12] or representative corollaries thereof, as they are applicable to our situation. Consider the eigenvalue problem in the form [6]

$$JLv = \lambda v. \tag{32}$$

Let k_r denotes the number of positive eigenvalues of (32), counting multiplicity and k_c be the number of complex valued eigenvalues with positive real part. Assuming $\Im L = 0$, we have that the complex eigenvalues come in pairs $\lambda, \bar{\lambda}$ and hence, k_c is an even integer. Finally, we need to introduce the Krein signature of eigenvalues lying on the imaginary axes $i\mathbf{R}$ as follows. For an eigenvalue $\lambda \in i\mathbf{R}$, denote the eigenspace by X_λ. The negative Krein index of λ is defined by[7]

$$k_i^-(\lambda) := n(\langle P_{X_\lambda} L P_{X_\lambda} u, u\rangle \, ,$$

that is the number of negative eigenvalues of the quadratic form defined above[8]. The total negative Krein index is then

$$k_i^- = \sum_{\lambda \in i\mathbf{R}} k_i^-(\lambda).$$

The sum of these last three quantities is called a Hamiltonian index for the eigenvalue problem (32), namely

$$K_{Ham} := k_r + k_i + k_c.$$

Next, assuming that $Ker(L)$ is finite dimensional, let $Ker(L) = span\{\psi_j : j = 1, \ldots, N\}$, where $\{\psi_j\}_{j=1}^N$ are linearly independent. Assume that there are linearly independent vectors

[6]Here, we insist that $L = L^*, J^* = -J$ and in addition, a number of additional technical conditions need to be imposed, but they will be all satisfied under our assumptions.

[7]$P_{X_\lambda} : L^2 \to L^2$ stands for the orthogonal projection over X_λ.

[8]Hereafter $n(L)$ for a self-adjoint operator/matrix will be used to denote the number of negative e-values.

$\{\phi_j\}_{j=1}^N$, so that $JL\phi_j = \psi_j$. Introduce $D = (D_{ij})_{i,j=1}^N$ and $D_{ij} = \langle \phi_i, L\phi_j\rangle$ and we also require that D is invertible.

In other words, the assumptions are that $Ker(JL)$ has geometric multiplicity N and algebraic multiplicity $2N$ and moreover, each eigenvector comes with exactly one generalized eigenvector. Equivalently $P_{X_\lambda} JLP_{X_\lambda}$ can be represented as a matrix with N Jordan cells, each of dimension two.

Under these assumptions, the main result of the theory is the following index counting formula

$$K_{Ham} = n(L) - n(D). \tag{33}$$

In the particular case, $N = 1$, which will be of importance to us, we have
$D = \langle L^{-1}[J^{-1}\psi_0], J^{-1}\psi_0\rangle$, where ψ_0 is the only vector in $Ker(L)$. Hence, (33) now reads

$$k_r + k_i + k_c = n(L) - n(\langle L^{-1}[J^{-1}\psi_0], J^{-1}\psi_0\rangle). \tag{34}$$

We also need some basic Fourier analysis. Define the Fourier transform and its inverse via

$$\hat{f}(\xi) = \int f(x)e^{-2\pi i x\xi}dx, f(x) = \int \hat{f}(\xi)e^{2\pi i x\xi}dx.$$

Consequently, we may define the operators $\sqrt{-\partial_x^2}$ or more generally, $(-\partial_x^2 + \epsilon^2)^{1/2}$ simply by multiplication on the Fourier side

$$\mathcal{F}[\sqrt{-\partial_x^2} f](\xi) = 2\pi|\xi|\hat{f}(\xi), \mathcal{F}[(-\partial_x^2 + \epsilon^2)^{1/2} f](\xi) = (4\pi^2\xi^2 + \epsilon^2)^{1/2}\hat{f}(\xi).$$

As we have explained above, assume that (17) holds and yet, there is a (complex) instability, say $\lambda_0 \notin i\mathbf{R}$. That is, there is a function u_0, so that

$$(\mathcal{L} - \lambda_0\partial_x)u_0 = 0.$$

Consider the the orthonormal systems $\{\psi_1, \ldots, \psi_n\}$ and $\{\psi_1^*, \ldots, \psi_n^*\}$ that provide basis for $Ker(\mathcal{L} - \lambda_0\partial_x)$ and $Ker(\mathcal{L} + \bar{\lambda}_0\partial_x)$ respectively. We consider approximate eigenvalue problems as follows

$$(\mathcal{L} + \sum_{j=1}^n \kappa_j L_j - \lambda_0(-\partial_x^2 + \epsilon^2)^{1/2}\mathcal{J})u = 0, \tag{35}$$

where $\mathcal{J} = -\mathcal{J}^*$ is the Hilbert transform (i.e. $\mathcal{J}\sqrt{-\partial_x^2} = \partial_x$ or $\widehat{\mathcal{J}f}(\xi) = isgn(\xi)\hat{f}(\xi)$), κ_j are real parameters and $L_j : L_j f = \langle f, h_j\rangle h_j$ are rank one projections.

Lemma 5. *There exists vectors* $\{h_j\}_{j=1}^n \in L^2(\mathbf{R}^1)$, *so that*

$$\langle h_j, \psi_0\rangle = 0,\ \langle h_j, \psi_i^*\rangle = \delta_{ij},\ \langle h_j, \psi_1\rangle = 1.$$

Proof. We will construct $h_j = h_j^1 + h_j^2$, where

$$\langle h_j^1, \psi_0\rangle = 0,\ \langle h_j^1, \psi_i^*\rangle = \delta_{ij},\ \langle h_j^1, \psi_1\rangle = 0$$

and

$$\langle h_j^2, \psi_0\rangle = 0,\ \langle h_j^2, \psi_i^*\rangle = 0,\ \langle h_j^2, \psi_1\rangle = 1.$$

For h_j^1, we wish to construct a vector, so that $h_j^1 \perp span\{\psi_0, \psi_1^*, \ldots, \psi_{j-1}^*, \psi_{j+1}^*, \ldots, \psi_n^*, \psi_1\}$ and $\langle h_j^1, \psi_j^* \rangle = 1$. This is possible, only if $\psi_j^* \notin span\{\psi_0, \psi_1^*, \ldots, \psi_{j-1}^*, \psi_{j+1}^*, \ldots, \psi_n^*, \psi_1\}$, which we now verify. Suppose (for a contradiction)

$$\psi_j^* = a_0\psi_0 + \sum_{i\neq j} c_i\psi_i^* + b_0\psi_1.$$

Applying $\mathcal{L} + \bar{\lambda}_0\partial_x$ to both sides yields

$$0 = a_0(\mathcal{L} + \bar{\lambda}_0\partial_x)(\psi_0) + b_0(\mathcal{L} + \bar{\lambda}_0\partial_x)(\psi_1) = a_0\bar{\lambda}_0\psi_0' + b_0(\bar{\lambda}_0 + \lambda_0)\psi_1'.$$

From this, since both ψ_0, ψ_1 are localized, it follows that $a_0\bar{\lambda}_0\psi_0 + b_0(\bar{\lambda}_0 + \lambda_0)\psi_1 = 0$. Taking $\mathcal{L}$ on both sides of the last identity yields $b_0\lambda_0(\bar{\lambda}_0 + \lambda_0)\psi_1' = 0$ and since $\psi_1' \neq 0$, $\lambda_0(\bar{\lambda}_0 + \lambda_0) \neq 0$ (recall $\lambda_0 \notin i\mathbf{R}$), it follows that $b_0 = 0$, whence $a_0 = 0$. But then,

$$\psi_j^* = \sum_{i\neq j} c_i\psi_i^*,$$

is contradictory , since $span\{\psi_1^*, \ldots, \psi_n^*\}$ is an orthonormal system. Thus, we have shown that $\psi_j^* \notin span\{\psi_0, \psi_1^*, \ldots, \psi_{j-1}^*, \psi_{j+1}^*, \ldots, \psi_n^*, \psi_1\}$.

It is now easy to produce h_j^1. Indeed, denoting the orthogonal projection $P_j : L^2 \to span\{\psi_0, \psi_1^*, \ldots, \psi_{j-1}^*, \psi_{j+1}^*, \ldots, \psi_n^*, \psi_1\}^\perp$, set $h_j^1 := \alpha_j P_j\psi_j^*$, where α_j is a non-zero scalar to be determined momentarily. Note that $P_j\psi_j^* \neq 0$, since $\psi_j^* \notin span\{\psi_0, \psi_1^*, \ldots, \psi_{j-1}^*, \psi_{j+1}^*, \ldots, \psi_n^*, \psi_1\}$. We have by construction $\langle h_j^1, \psi_0 \rangle = 0$, $\langle h_j^1, \psi_1 \rangle = 0 = \langle h_j^1, \psi_i^* \rangle, i \neq j$. Also, since $P_j^2 = P_j$,

$$\langle h_j^1, \psi_j^* \rangle = \alpha_j \langle P_j\psi_j^*, \psi_j^* \rangle = \alpha_j \langle P_j^2\psi_j^*, \psi_j^* \rangle = \alpha_j\|P_j\psi_j^*\|^2 = 1,$$

if we select $\alpha_j := \|P_j\psi_j^*\|^{-2}$, which is possible, since $P_j\psi_j^* \neq 0$.

The construction of h_j^2 is similar. We need to check $\psi_1 \notin span\{\psi_0, \psi_1^*, \ldots, \psi_n^*\}$. Suppose (for a contradiction)

$$\psi_1 = a_0\psi_0 + \sum_{j=1}^n c_j\psi_j^*.$$

Take again $\mathcal{L} + \bar{\lambda}_0\partial_x$ on both sides to obtain $(\mathcal{L} + \bar{\lambda}_0\partial_x)\psi_1 - a_0(\mathcal{L} + \bar{\lambda}_0\partial_x)\psi_0 = 0$. But the last identity is equivalent to $(\lambda_0 + \bar{\lambda}_0)\psi_1' - a_0\bar{\lambda}_0\psi_0' = 0$. Again, by the fact that ψ_1, ψ_0 vanish at infinity, it follows that $(\lambda_0 + \bar{\lambda}_0)\psi_1 - a_0\bar{\lambda}_0\psi_0 = 0$. Taking $\mathcal{L}$ then implies $\lambda_0(\lambda_0 + \bar{\lambda}_0)\psi_1' = 0$. Again, $\lambda_0(\bar{\lambda}_0 + \lambda_0) \neq 0$, whence $\psi_1' = 0$, a contradiction. This shows that $\psi_1 \notin span\{\psi_0, \psi_1^*, \ldots, \psi_n^*\}$.

Denoting the orthogonal projection $R_j : L^2 \to span\{\psi_0, \psi_1^*, \ldots, \psi_n^*\}^\perp$ and $h_j^2 := \beta_j R_j\psi_1$, we have that $\langle h_j^2, \psi_0 \rangle = 0$, $\langle h_j^2, \psi_i^* \rangle = 0, i = 1, \ldots, n$ and $R_j\psi_1 \neq 0$, since $\psi_1 \notin span\{\psi_0, \psi_1^*, \ldots, \psi_n^*\}$. Moreover $R_j^2 = R_j$ and

$$\langle h_j^2, \psi_1 \rangle = \beta_j \langle R_j\psi_1, \psi_1 \rangle = \beta_j \langle R_j^2\psi_1, \psi_1 \rangle = \beta_j\|R_j\psi_1\|^2 = 1,$$

if $\beta_j := \|R_j\psi_1\|^{-2}$.

□

3.2. **The Lyapunov-Schmidt reduction.** By our assumption, (35) has solution for $\epsilon = 0 = \kappa_1 = \ldots = \kappa_n$, namely $u_0 \in span\{\psi_1, \ldots, \psi_n\}$. We now set out to construct a solution for all values of $0 < \epsilon << 1$. Set

$$G(\epsilon; \kappa_1, \ldots, \kappa_n, f) = (\mathcal{L} + \sum_{j=1}^n \kappa_j L_j - \lambda_0(-\partial_x^2 + \epsilon^2)^{1/2} \mathcal{J})[\psi_1 + f], \tag{36}$$

where f will be an element of $span\{\psi_1, \ldots, \psi_n\}^\perp$. We will apply the Lyapunov-Schmidt reduction method to show the existence of solutions in the neighbourhood of the point $\epsilon \sim 0$. Introduce the projection Q onto $span\{\psi_1^*, \ldots, \psi_n^*\}^\perp$, that is

$$Q[\chi] = \chi - \sum_{j=1}^n \langle \chi, \psi_j^* \rangle \psi_j^*.$$

Consider first the equation

$$Q[G(\epsilon; \kappa_1, \ldots, \kappa_n, f)] = Q[(\mathcal{L} + \sum_{j=1}^n \kappa_j L_j - \lambda_0(-\partial_x^2 + \epsilon^2)^{1/2} \mathcal{J})[\psi_1 + f]] = 0, \tag{37}$$

where f is the unknown function in $span\{\psi_1, \ldots, \psi_n\}^\perp$ and $\epsilon, \kappa_1, \ldots, \kappa_n$ are considered parameters close to the base point $(0; 0, \ldots, 0)$. By the implicit function theorem, (37) will have a solution $f = f(\epsilon; \kappa_1, \ldots, \kappa_n)$ if

$$\left\langle D_f(0; 0, \ldots, 0), \tilde{f} \right\rangle = Q[(\mathcal{L} - \lambda_0 \partial_x)(\tilde{f})]$$

is a bijection from $span\{\psi_1, \ldots, \psi_n\}^\perp$ to $Ran[Q] = span\{\psi_1^*, \ldots, \psi_n^*\}^\perp$. But this is exactly the requirement of (14). Indeed, according to (14) for every $g \in span\{\psi_1^*, \ldots, \psi_n^*\}^\perp$, there is an unique $\tilde{f} \in span\{\psi_1, \ldots, \psi_n\}^\perp$, so that $(\mathcal{L} - \lambda_0 \partial_x)\tilde{f} = g$. Thus, we have shown the existence of C^1 function $f = f(\epsilon; \kappa_1, \ldots, \kappa_n) : f(0; 0, \ldots, 0) = 0$, which solves (37).

In the second stage of the Lyapunov-Schmidt reduction procedure, it remains to resolve the n remaining directions, namely $\psi_1^*, \ldots, \psi_n^*$. More precisely, we need to solve the $n \times n$ system

$$H_i(\epsilon; \vec{\kappa}) = \left\langle (\mathcal{L} + \sum_{j=1}^n \kappa_j L_j - \lambda_0(-\partial_x^2 + \epsilon^2)^{1/2} \mathcal{J})(\psi_1 + f(\epsilon; \vec{\kappa})), \psi_i^* \right\rangle = 0, i = 1, \ldots, n,$$

with unknowns $\vec{k} = (\kappa_1, \ldots, \kappa_n)$ in terms of ϵ. Again, this will be done by the implicit function theorem, close to the base point $\epsilon \sim 0$, $\vec{k} \sim (0, \ldots, 0)$. This is now a finite dimensional system, we just need to make sure that the Jacobian map is non-singular at the base point, that is

$$\det \left(\frac{\partial H_i}{\partial \kappa_j} \right) (0; 0, \ldots, 0) \neq 0.$$

We have

$$\frac{\partial H_i}{\partial \kappa_j}(0; 0, \ldots, 0) = \left\langle (\mathcal{L} - \lambda_0 \partial_x)[\frac{\partial f}{\partial_{\kappa_j}}], \psi_i^* \right\rangle + \langle L_j \psi_1, \psi_i^* \rangle = \langle L_j \psi_1, \psi_i^* \rangle,$$

since $\left\langle (\mathcal{L} - \lambda_0 \partial_x)[\frac{\partial f}{\partial \kappa_j}], \psi_i^* \right\rangle = \left\langle \frac{\partial f}{\partial \kappa_j}, (\mathcal{L} + \bar{\lambda}_0 \partial_x)[\psi_i^*] \right\rangle = 0$. Finally,

$$\langle L_j \psi_1, \psi_i^* \rangle = \langle \psi_1, h_j \rangle \langle \psi_i^*, h_j \rangle = \delta_{ij},$$

according to Lemma 5. Thus, $\left(\frac{\partial H_i}{\partial \kappa_j}\right)(0; 0, \ldots, 0) = Id$, and hence is not singular. We have proved that (36) has C^1 solutions $\kappa_1(\epsilon), \ldots, \kappa_n(\epsilon)$ and $f(\epsilon) = f(\epsilon; \kappa_1(\epsilon), \ldots, \kappa_n(\epsilon))$, so that $\kappa_1(0) = \ldots = \kappa_n(0) = 0$ and $f(0) = 0$. We formulate the result in the following

Proposition 4. *There exists $\epsilon_0 > 0$ and C^1 functions $\kappa_1(\epsilon), \ldots, \kappa_n(\epsilon) : \lim_{\epsilon \to 0} \kappa_j(\epsilon) = 0$ and $f(\epsilon) \in D(\mathcal{L})$, defined in $\epsilon : |\epsilon| < \epsilon_0$, so that*

$$(\mathcal{L} + \sum_{j=1}^{n} \kappa_j(\epsilon) L_j - \lambda_0 (-\partial_x^2 + \epsilon^2)^{1/2} \mathcal{J})[\psi_1 + f(\epsilon)] = 0 \tag{38}$$

3.3. **Some auxiliary spectral results.** Now that we have established that the operators $\mathcal{L} + \sum_{j=1}^{n} \kappa_j(\epsilon) L_j - \lambda_0(-\partial_x^2 + \epsilon^2)^{1/2} \mathcal{J}$ have instability for all small ϵ, we proceed to establish some spectral properties for the self-adjoint operator $L_{\vec{\kappa}} := \mathcal{L} + \sum_{j=1}^{n} \kappa_j(\epsilon) L_j$ for $|\vec{k}| << 1$, which will allows us to apply the Kapitula-Kevrekidis-Sandstede theory to its index.

Lemma 6. *There exists $\kappa_0 > 0$, so that the self-adjoint operator $L_{\vec{\kappa}}$, has exactly one negative simple eigenvalue, a simple eigenvalue at zero, with eigenvector ψ_0 and the rest of the spectrum is strictly positive. In fact,*

$$\sigma(\mathcal{L}_{\vec{\kappa}}) = \{-\sigma^2 + O(\kappa)\} \cup \{0\} \cup \sigma_+(\mathcal{L}), \sigma_+(\mathcal{L}) \subset (\delta^2 + O(\kappa), \infty),$$

Proof. We use the Rayleigh principle for the eigenvalues. For the lowest eigenvalue, we have

$$\sigma_0(\mathcal{L}_{\vec{\kappa}}) = \inf_{\|f\|=1} \langle \mathcal{L}_{\vec{\kappa}} f, f \rangle = \inf_{\|f\|=1} (\langle \mathcal{L} f, f \rangle - \sum_{j=1}^{n} \kappa_j \langle L_j f, f \rangle) \leq -\sigma^2 + O(\kappa),$$

which is negative for all small enough $|\vec{\kappa}|$. Thus, there is a negative eigenvalue, with say an eigenvector $f_{\vec{\kappa}}$. Next, since $\mathcal{L}_{\vec{\kappa}} \psi_0 = 0$ by construction (and thus $\psi_0 \perp f_{\vec{\kappa}}$), we have that

$$\sigma_1(\mathcal{L}_{\vec{\kappa}}) = \inf_{\|f\|=1 : f \perp f_{\vec{\kappa}}} \langle \mathcal{L}_{\vec{\kappa}} f, f \rangle \leq \langle \mathcal{L}_{\vec{\kappa}} \psi_0, \psi_0 \rangle = 0.$$

Finally,

$$\sigma_2(\mathcal{L}_{\vec{\kappa}}) \geq \inf_{\|f\|=1 : f \perp f_0, \psi_0} \langle \mathcal{L}_{\vec{\kappa}} f, f \rangle = \inf_{\|f\|=1 : f \perp f_0, \psi_0} (\langle \mathcal{L} f, f \rangle - \sum_{j=1}^{n} \kappa_j \langle L_j f, f \rangle) \geq \delta^2 + O(\kappa),$$

which is positive for all small enough $|\vec{\kappa}|$. It follows that $\sigma_0(\mathcal{L}_{\vec{\kappa}}) < 0, \sigma_1(\mathcal{L}_{\vec{\kappa}}) = 0$, and the rest of the spectrum is strictly positive. □

We now rewrite (38). Let $z_\epsilon := (-\partial_x^2 + \epsilon^2)^{1/4} [\psi_1 + f(\epsilon)]$ and thus,

$$\mathcal{L}_{\vec{\kappa(\epsilon)}} (-\partial_x^2 + \epsilon^2)^{-1/4} z_\epsilon = \lambda_0 (-\partial_x^2 + \epsilon^2)^{1/4} \mathcal{J} z_\epsilon.$$

Takin $(-\partial_x^2 + \epsilon^2)^{-1/4}$ on both sides of the last identity yields

$$(-\partial_x^2 + \epsilon^2)^{-1/4} \mathcal{L}_{\vec{\kappa(\epsilon)}} (-\partial_x^2 + \epsilon^2)^{-1/4} z_\epsilon = \lambda_0 \mathcal{J} z_\epsilon.$$

Denoting $\mathcal{L}_\epsilon^\diamond := (-\partial_x^2+\epsilon^2)^{-1/4}\mathcal{L}_{\vec{\kappa(\epsilon)}}(-\partial_x^2+\epsilon^2)^{-1/4}$ allows us to finally rewrite

$$\mathcal{L}_\epsilon^\diamond z_\epsilon = \lambda_0 \mathcal{J} z_\epsilon. \tag{39}$$

Here the operator $\mathcal{L}_\epsilon^\diamond$ can be constructed by the Friedrich's extension corresponding to the form

$$q_\epsilon(f,g) = \langle \mathcal{L}_\epsilon^\diamond f, g\rangle = \left\langle \mathcal{L}_{\kappa(\epsilon)}(-\partial_x^2+\epsilon^2)^{-1/4} f, (-\partial_x^2+\epsilon^2)^{-1/4} g\right\rangle,$$

which is well-defined $q_\epsilon : H^{s/2+1/2}\times H^{s/2+1/2}$. Thus, $\mathcal{L}_\epsilon^\diamond$ is self-adjoint, with domain $D(\mathcal{L}_\epsilon^\diamond) = H^{s+1}$. We have the following result concerning its spectrum.

Lemma 7. *There exists $\epsilon_0 > 0$, so that for all $\epsilon : |\epsilon| < \epsilon_0$, $\mathcal{L}_\epsilon^\diamond$ has one simple negative eigenvalue, a simple eigenvalue at zero, with an eigenvector $(-\partial_x^2+\epsilon^2)^{1/4}\psi_0$ and all other eigenvalues (if any) are strictly positive.*

Regarding the essential spectrum, it is contained in $(0,\infty)$. More specifically, there exists $\delta_\epsilon > 0$, so that

$$\sigma_{ess}(\mathcal{L}_\epsilon^\diamond) \subset [\delta_\epsilon, \infty).$$

Proof. Recall, $f_\epsilon = f_{\vec{\kappa(\epsilon)}} : \|f_\epsilon\| = 1$ is the normalized negative eigenvector corresponding to $\mathcal{L}_{\vec{\kappa}}$. We have

$$\begin{aligned}\sigma_0(\mathcal{L}_\epsilon^\diamond) &= \inf_{\|f\|=1}\langle \mathcal{L}_\epsilon^\diamond f, f\rangle \leq \frac{\left\langle \mathcal{L}_\epsilon^\diamond(-\partial_x^2+\epsilon^2)^{1/4} f_\epsilon, (-\partial_x^2+\epsilon^2)^{1/4} f_\epsilon\right\rangle}{\|(-\partial_x^2+\epsilon^2)^{1/4} f_\epsilon\|^2} = \\ &= \frac{\left\langle \mathcal{L}_{\kappa(\epsilon)} f_\epsilon, f_\epsilon\right\rangle}{\|(-\partial_x^2+\epsilon^2)^{1/4} f_\epsilon\|^2} < 0,\end{aligned}$$

for all small enough ϵ, by Lemma 6.

On one hand, $\mathcal{L}_\epsilon^\diamond[(-\partial_x^2+\epsilon^2)^{1/4}\psi_0] = (-\partial_x^2+\epsilon^2)^{-1/4}\mathcal{L}_{\vec{\kappa(\epsilon)}}\psi_0 = 0$, so zero is an eigenvalue. On the other,

$$\begin{aligned}\sigma_1(\mathcal{L}_\epsilon^\diamond) &\geq \inf_{f\neq 0: f\perp (-\partial_x^2+\epsilon^2)^{-1/4} f_\epsilon} \langle \mathcal{L}_\epsilon^\diamond f, f\rangle = \\ &= \inf_{f\neq 0: (-\partial_x^2+\epsilon^2)^{-1/4} f\perp f_\epsilon} \left\langle \mathcal{L}_{\vec{\kappa(\epsilon)}}(-\partial_x^2+\epsilon^2)^{-1/4} f, (-\partial_x^2+\epsilon^2)^{-1/4} f\right\rangle \geq 0,\end{aligned}$$

since $\sigma_1(\mathcal{L}_{\vec{\kappa(\epsilon)}}) = \inf_{g\neq 0: g\perp f_\epsilon} \frac{\left\langle \mathcal{L}_{\vec{\kappa(\epsilon)}} g, g\right\rangle}{\|g\|^2} = 0$. These last two statements imply $\sigma_1(\mathcal{L}_\epsilon^\diamond) = 0$.

Finally, since $\sigma_2(\mathcal{L}_\epsilon^\diamond) \geq \sigma_1(\mathcal{L}_\epsilon^\diamond) = 0$, we have two options - either $\sigma_2(\mathcal{L}_\epsilon^\diamond) > 0$ for all sufficiently small ϵ or $\sigma_2(\mathcal{L}_\epsilon^\diamond) = 0$ for some sequence $\epsilon_n \to 0+$. We show that the latter assertion cannot hold. Indeed, suppose that $\sigma_2(\mathcal{L}_\epsilon^\diamond) = 0$ for some small ϵ. Then zero is an eigenvalue with multiplicity two for $\mathcal{L}_\epsilon^\diamond$. But a simple analysis of the equation $\mathcal{L}_\epsilon^\diamond z = 0$ shows that in this case, there are two linearly independent functions z_1, z_2, so that

$$\mathcal{L}_{\vec{\kappa(\epsilon)}}[(-\partial_x^2+\epsilon^2)^{-1/4} z_1] = 0 = \mathcal{L}_{\vec{\kappa(\epsilon)}}[(-\partial_x^2+\epsilon^2)^{-1/4} z_2].$$

But we already know that for all ϵ small enough, zero is a simple eigenvalue for $\mathcal{L}_{\vec{\kappa(\epsilon)}}$, with eigenvalue ψ_0. It follows that $(-\partial_x^2+\epsilon^2)^{-1/4} z_j = \psi_0, j = 1,2$. Thus, $z_1 = z_2 = (-\partial_x^2+\epsilon^2)^{1/4}\psi_0$, a contradiction. It follows that the rest of the spectrum is strictly positive.

For the essential spectrum calculations, we make an extensive use of the assumption (16) and follow the method of the proof of Proposition 4.2, [14]. Define

$$\tilde{\mathcal{L}}_\epsilon = (-\partial_x^2+\epsilon^2)^{-1/4}\mathcal{L}_0(-\partial_x^2+\epsilon^2)^{-1/4}, \tilde{\mathcal{K}}_\epsilon = (-\partial_x^2+\epsilon^2)^{-1/4}(\mathcal{K}+\sum_{j=1}^n \kappa_j(\epsilon)L_j)(-\partial_x^2+\epsilon^2)^{-1/4}$$

so that $\mathcal{L}_{\vec{\kappa(\epsilon)}} = \tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon$. We first compute the essential spectrum of $\tilde{\mathcal{L}}_\epsilon$. This is easy to do, as the operator is given by a multiplier and hence, the essential spectrum is just the range of the multiplier[9]. Since

$$\widehat{\tilde{\mathcal{L}}_\epsilon f}(\xi) = \frac{q_0(\xi)}{(4\pi^2\xi^2+\epsilon^2)^{1/2}}.$$

Note that by (16), the function $\frac{q_0(\xi)}{(4\pi^2\xi^2+\epsilon^2)^{1/2}}$ is strictly positive on compact sets of $\mathbf{R}$ and

$$\lim_{|\xi|\to\infty} \frac{q_0(\xi)}{(4\pi^2\xi^2+\epsilon^2)^{1/2}} = \lim_{|\xi|\to\infty} \frac{q_0(\xi)}{|\xi|^s}\frac{|\xi|^s}{(4\pi^2\xi^2+\epsilon^2)^{1/2}} = \infty,$$

by (16) and since $s > 1$. It follows that it achieves a minimum $\delta_\epsilon > 0$. Thus,

$$\sigma_{ess}(\tilde{\mathcal{L}}_\epsilon) = Range\left[\xi \to \frac{q_0(\xi)}{(4\pi^2\xi^2+\epsilon^2)^{1/2}}\right] \subset [\delta_\epsilon, \infty).$$

We now need to show that $\sigma_{ess}(\mathcal{L}_{\vec{\kappa(\epsilon)}}) = \sigma_{ess}(\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon) = \sigma_{ess}(\tilde{\mathcal{L}}_\epsilon)$, which follows from Corollary 1, p. 113, [18], if we can show that $(\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon + i)^{-1} - (\tilde{\mathcal{L}}_\epsilon + i)^{-1}$ is a compact operator. To that end, write by the resolvent identity

$$(\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon + i)^{-1} - (\tilde{\mathcal{L}}_\epsilon + i)^{-1} = -(\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon + i)^{-1}\tilde{\mathcal{K}}_\epsilon(\tilde{\mathcal{L}}_\epsilon + i)^{-1}. \tag{40}$$

The operators on the sides can also be represented via the resolvent identity as follows

$$\begin{aligned} (\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon + i)^{-1} &= \tilde{\mathcal{L}}_\epsilon^{-1} - (\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon + i)^{-1}(\tilde{\mathcal{K}}_\epsilon + i)\tilde{\mathcal{L}}_\epsilon^{-1} \\ (\tilde{\mathcal{L}}_\epsilon + i)^{-1} &= \tilde{\mathcal{L}}_\epsilon^{-1} + i\tilde{\mathcal{L}}_\epsilon^{-1}(\tilde{\mathcal{L}}_\epsilon + i)^{-1} \end{aligned}$$

Plugging these two formulas into (40) yields

$$\begin{aligned} &(\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon + i)^{-1} - (\tilde{\mathcal{L}}_\epsilon + i)^{-1} = \\ = \; &[(\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon + i)^{-1}(\tilde{\mathcal{K}}_\epsilon + i)\tilde{\mathcal{L}}_\epsilon^{-1} - \tilde{\mathcal{L}}_\epsilon^{-1}]\tilde{\mathcal{K}}_\epsilon[\tilde{\mathcal{L}}_\epsilon^{-1} + i\tilde{\mathcal{L}}_\epsilon^{-1}(\tilde{\mathcal{L}}_\epsilon + i)^{-1}]. \end{aligned}$$

Note that all terms above are in the form of a bounded operator times $\tilde{\mathcal{L}}_\epsilon^{-1}\tilde{\mathcal{K}}_\epsilon\tilde{\mathcal{L}}_\epsilon^{-1}$. But now[10]

$$\tilde{\mathcal{L}}_\epsilon^{-1}\tilde{\mathcal{K}}_\epsilon\tilde{\mathcal{L}}_\epsilon^{-1} = (-\partial_x^2+\epsilon^2)^{1/4}\mathcal{L}_0^{-1}[K + \sum_{j=1}^n \kappa_j(\epsilon)L_j](-\partial_x^2+\epsilon^2)^{1/4}\mathcal{L}_0^{-1}.$$

By our assumption (16), this last operator is compact and hence Lemma 7 is established. □

[9]Here the multiplier is viewed as a bounded function of ξ

[10]Note that $\mathcal{L}_0$ commutes with $(-\partial_x^2+\epsilon^2)^{1/4}$

The last spectral result that we need, before applying the Kapitula-Kevrekidis-Sandstede theory is regarding the generalized kernel of $\mathcal{J}\mathcal{L}_\epsilon^\diamond$. This is clearly relevant because of the form of the spectral problem (39) and the unitarity property of the Hilbert transform: $\mathcal{J}^{-1} = \mathcal{J}^* = -\mathcal{J}$.

Lemma 8. *Under our assumption* (17),

$$gKer(\mathcal{J}\mathcal{L}_\epsilon^\diamond) = span\{(-\partial_x^2+\epsilon^2)^{1/4}\psi_0, (-\partial_x^2+\epsilon^2)^{1/4}\mathcal{L}_{\vec{\kappa(\epsilon)}}^{-1}\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0\}.$$

Proof. First, $(-\partial_x^2+\epsilon^2)^{1/4}\psi_0 \in Ker(\mathcal{J}\mathcal{L}_\epsilon^\diamond)$ by a direct verification. Next, to determine the next element in $gKer$, we need to solve

$$\mathcal{J}\mathcal{L}_\epsilon^\diamond f = (-\partial_x^2+\epsilon^2)^{1/4}\psi_0.$$

Applying $\mathcal{J}^{-1} = -\mathcal{J}$ on both sides leads[11] to

$$(-\partial_x^2+\epsilon^2)^{-1/4}\mathcal{L}_{\vec{\kappa(\epsilon)}}(-\partial_x^2+\epsilon^2)^{-1/4}f = \mathcal{L}_\epsilon^\diamond f = -\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/4}\psi_0$$

Applying $(-\partial_x^2+\epsilon^2)^{1/4}$ on both sides yields

$$\mathcal{L}_{\vec{\kappa(\epsilon)}}[(-\partial_x^2+\epsilon^2)^{-1/4}f] = -\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0.$$

By the Fredholm property (which holds for $\mathcal{L}$ and so for $\mathcal{L}_{\vec{\kappa(\epsilon)}}$, since it is a finite rank perturbation of $\mathcal{L}$), this last equation has solution if and only if $\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0 \perp \psi_0$. But this is indeed satisfied, since $\mathcal{J}$ is anti-symmetric and

$$\left\langle \mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0, \psi_0\right\rangle = \left\langle \mathcal{J}(-\partial_x^2+\epsilon^2)^{1/4}\psi_0, (-\partial_x^2+\epsilon^2)^{1/4}\psi_0\right\rangle = 0.$$

Thus,

$$f = -(-\partial_x^2+\epsilon^2)^{1/4}\mathcal{L}_{\vec{\kappa(\epsilon)}}^{-1}[\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0] \in gKer(\mathcal{J}\mathcal{L}_\epsilon^\diamond).$$

To see there are no other elements of this Jordan cell, we need to consider the equation

$$\mathcal{J}\mathcal{L}_\epsilon^\diamond g = -(-\partial_x^2+\epsilon^2)^{1/4}\mathcal{L}_{\vec{\kappa(\epsilon)}}^{-1}[\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0] \in gKer(\mathcal{J}\mathcal{L}_\epsilon^\diamond),$$

which after applying $\mathcal{J}^{-1}$ reduces to

$$\mathcal{L}_\epsilon^\diamond g = \mathcal{J}(-\partial_x^2+\epsilon^2)^{1/4}\mathcal{L}_{\vec{\kappa(\epsilon)}}^{-1}[\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0] \in gKer(\mathcal{J}\mathcal{L}_\epsilon^\diamond)$$

For the existence of solution, the Fredholm solvability condition now requires

$$\left\langle \mathcal{J}(-\partial_x^2+\epsilon^2)^{1/4}\mathcal{L}_{\vec{\kappa(\epsilon)}}^{-1}[\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0], (-\partial_x^2+\epsilon^2)^{1/4}\psi_0\right\rangle = 0$$

On the other hand,

$$\left\langle \mathcal{J}(-\partial_x^2+\epsilon^2)^{1/4}\mathcal{L}_{\vec{\kappa(\epsilon)}}^{-1}[\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0], (-\partial_x^2+\epsilon^2)^{1/4}\psi_0\right\rangle =$$
$$= -\left\langle \mathcal{L}_{\vec{\kappa(\epsilon)}}^{-1}\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0, \mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0\right\rangle$$

However, we claim that

$$\lim_{\epsilon\to 0}\left\langle \mathcal{L}_{\vec{\kappa(\epsilon)}}^{-1}\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0, \mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0\right\rangle = \left\langle \mathcal{L}^{-1}\psi_0', \psi_0'\right\rangle \tag{41}$$

[11]Note $\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/4} = (-\partial_x^2+\epsilon^2)^{1/4}\mathcal{J}$

If we show (41) and since $\langle \mathcal{L}^{-1}\psi_0', \psi_0'\rangle \neq 0$, it follows that for all small enough ϵ, $\left\langle \mathcal{L}^{-1}_{\vec{\kappa(\epsilon)}} \mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0, \mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0 \right\rangle \neq 0$ and hence the Fredholm solvability condition fails, hence the Jordan cell is only of length two.

Proof of (41)**:**
We see first that $\mathcal{L}^{-1}\sum_j \kappa_j L_j$ is a well-defined, since $Ran[L_j] = span\{h_j\} \subset \{\psi_0\}^\perp$. Hence,

$$\mathcal{L}_{\vec{\kappa}}^{-1}|_{\{\psi_0\}^\perp} = (\mathcal{L}(Id + \mathcal{L}^{-1}\sum_{j=1}^n \kappa_j L_j))^{-1}|_{\{\psi_0\}^\perp} = \sum_{l=0}^\infty (-1)^l (\mathcal{L}^{-1}\sum_{j=1}^n \kappa_j L_j))^l \mathcal{L}^{-1}|_{\{\psi_0\}^\perp}$$

and hence $\|\mathcal{L}_{\vec{\kappa}}^{-1} - \mathcal{L}^{-1}\|_{B(L^2)} \to 0$, as $|\vec{\kappa}| \to 0$. Next, by Plancherel's

$$\|(-\partial_x^2+\epsilon^2)^{1/2}\psi_0 - \sqrt{-\partial_x^2}\psi_0\|_{L^2}^2 = \int \frac{\epsilon^4}{(\sqrt{4\pi^2\xi^2+\epsilon^2}+2\pi|\xi|)^2}|\widehat{\psi_0}(\xi)|^2 d\xi \leq \epsilon^3\|\psi_0\|_{L^2}^2.$$

Thus, noting that $\psi_0' = \mathcal{J}\sqrt{-\partial_x^2}\psi_0$,

$$\begin{aligned}
& |\left\langle \mathcal{L}^{-1}_{\vec{\kappa(\epsilon)}} \mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0, \mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0 \right\rangle - \left\langle \mathcal{L}^{-1}\psi_0', \psi_0'\right\rangle| \leq \\
& |\left\langle [\mathcal{L}^{-1}_{\vec{\kappa(\epsilon)}} - \mathcal{L}^{-1}]\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0, \mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0 \right\rangle| + \\
+ \quad & |\left\langle \mathcal{L}^{-1}\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0, \mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0 \right\rangle - \left\langle \mathcal{L}^{-1}\psi_0', \psi_0'\right\rangle| \\
\leq \quad & \|\mathcal{L}^{-1}_{\vec{\kappa(\epsilon)}} - \mathcal{L}^{-1}\|_{B(L^2)}\|\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0\|_{L^2}^2 + \\
+ \quad & 2\|\mathcal{L}^{-1}\|_{B(L^2)}\|(-\partial_x^2+\epsilon^2)^{1/2}\psi_0 - \sqrt{-\partial_x^2}\psi_0\|(\|(-\partial_x^2+\epsilon^2)^{1/2}\psi_0\| + \|\sqrt{-\partial_x^2}\psi_0\|)
\end{aligned}$$

By the convergence results established above, we may conclude (41) and thus the proof of Lemma 8 is complete. □

3.4. **Conclusion of the proof of Theorem 2.** After all the necessary preparations, we are finally ready to finish the proof of Theorem 2. We are looking at the eigenvalue problem (39), which has instability according to Proposition 4. Thus,

$$K_{Ham} = k_r + k_c + k_i \geq 1.$$

According to Lemma 7 however, for all small ϵ, $\mathcal{L}_\epsilon^\diamond$ has a simple negative eigenvalue, or $n(\mathcal{L}_\epsilon^\diamond) = 1$. By the KKS theory, more specifically (33),

$$1 \leq K_{Ham} = n(\mathcal{L}_\epsilon^\diamond) - n(D) = 1 - n(D) \leq 1.$$

It follows that $K_{Ham} = 1$ and $n(D) = 0$, which means that

$$\left\langle (\mathcal{L}_\epsilon^\diamond)^{-1}\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/4}\psi_0, (-\partial_x^2+\epsilon^2)^{1/4}\psi_0 \right\rangle \geq 0.$$

for all small enough ϵ. On the other hand,

$$\left\langle (\mathcal{L}_\epsilon^\diamond)^{-1}\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/4}\psi_0, (-\partial_x^2+\epsilon^2)^{1/4}\psi_0 \right\rangle = \left\langle \mathcal{L}^{-1}_{\vec{\kappa(\epsilon)}} \mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0, \mathcal{J}(-\partial_x^2+\epsilon^2)^{1/2}\psi_0 \right\rangle,$$

which according to (41) converges, as $\epsilon \to 0$, to $\langle \mathcal{L}^{-1}\psi_0', \psi_0'\rangle < 0$. Hence

$$\left\langle (\mathcal{L}_\epsilon^\diamond)^{-1}\mathcal{J}(-\partial_x^2+\epsilon^2)^{1/4}\psi_0, (-\partial_x^2+\epsilon^2)^{1/4}\psi_0 \right\rangle < 0$$

for all small enough ϵ. We have reached a contradiction, which completes the proof of Theorem 2.

4. Applications to the generalized Bullough-Dodd models

We start with a simple model, on which all hypothesis of the theory are easily verfiable.

4.1. **Toy model.** Consider the model

$$u_{tx} = -u_{xx} + au - bu^p \tag{42}$$

where $a, b > 0$ and p is an integer. Applying the travelling wave ansatz $u(t,x) = \varphi(x - ct)$ yields the following equation

$$-(1-c)\varphi'' + a\varphi - b\varphi^p = 0 \tag{43}$$

This equation has the following solution for each $c < 1$

$$\varphi(\xi) = \left(\frac{a(p+1)}{2b}\right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}}\left(\frac{\sqrt{a}(p-1)}{2\sqrt{b}\sqrt{1-c}}x\right).$$

We consider the corresponding linearization around φ. Namely, let $u(t,x) = \varphi(x-ct) + v(t, x - ct)$ and plug it in (42). Ignoring all non-linear in v terms yields the linearized problem

$$v_{tx} = -(1-c)v'' + av - bp\varphi^{p-1}v. \tag{44}$$

Converting this to an eigenvalue problem via $v(t,x) = e^{\lambda t}z$ puts us in the form $\lambda z' = \mathcal{L}z$, with the operator

$$\mathcal{L} := -(1-c)\partial_x^2 + a - bp\varphi^{p-1}.$$

The operator $\mathcal{L}$ satisfies (10), (11), identical to the verification in the next section. Thus, we can apply Theorem 1 to study the spectral instability of φ. More precisely, for the spectral problem (44), it suffices to compute the quantity $\langle \mathcal{L}^{-1}\psi_0', \psi_0'\rangle = \langle \mathcal{L}^{-1}\varphi'', \varphi''\rangle$. Taking a derivative in c in the defining equation (43) yields

$$\mathcal{L}[\partial_c\varphi] + \varphi'' = 0,$$

whence

$$\left\langle \mathcal{L}^{-1}\varphi'', \varphi''\right\rangle = -\langle \partial_c\varphi, \varphi''\rangle = \langle \partial_c\varphi', \varphi'\rangle = \frac{1}{2}\partial_c\|\varphi'\|^2.$$

But

$$\|\varphi'\|^2 = (1-c)^{-1/2}c_{a,b,p},$$

for some positive constant $c_{a,b,p}$. Hence,

$$\left\langle \mathcal{L}^{-1}\varphi'', \varphi''\right\rangle = (1-c)^{-3/2}C(a,b,p) > 0,$$

whence by Theorem 1, we have instability for these waves, for all values of $c < 1$, $a, b > 0$, $p > 1$.

4.2. **Generalized Bullough-Dodd models.** We now consider the problem for the stability of a travelling wave of (1). The corresponding linearized problems are set up, so that they lead to eigenvalue problems of the form (7).

More concretely, consider a travelling wave solution $\varphi(x+ct), c>0$, which satisfies

$$-c\varphi'' + a\varphi - f(\varphi) = 0. \tag{45}$$

We require that φ is a positive and bounded bell-shaped function (i.e. even and strictly decreasing in $(0,\infty)$), so that $\varphi' \in L^2(\mathbf{R}^1)$, $\lim_{\xi\to\pm\infty} f'(\varphi) = 0$, $f'(\varphi), f(\varphi) \in L^1(\mathbf{R}^1)$. Thus, φ will have a maximum at zero and hence the only zero of φ' is at $z=0$. By Sturm-Liouville theory, this implies that for the linearized operator

$$\mathcal{L} := -c\partial_{\xi\xi} + a - f'(\varphi)$$

has a simple eigenvalue at zero (with an eigenvector φ') and a simple negative eigenvalue, say $-\sigma^2$ and the corresponding eigenfunction, say f_0. Note that f_0 will be an even function, as a ground state of the even potential $f'(\varphi)$. Finally, the rest of the spectrum is strictly positive by the Weyl's criteria. Next, we verify (11). We work with functions $v \in P_{>0}[L^2]$ and h, so that

$$\mathcal{L}h - \lambda P_{>0} h' = v,$$

and we need to establish that $\|h\|_{H^1} \leq C(\lambda)\|v\|_{L^2}$. Note that by Proposition 1 (and more precisely (9)), we already know that $\|h\|_{L^2} \leq const.\|v\|_{L^2}$. The defining equation for h is

$$\begin{aligned}(-c\partial_\xi^2 - \lambda\partial_\xi + a)h &= v + f'(\varphi)h + \lambda(\langle h', f_0\rangle f_0 + \langle h', \psi_0\rangle \psi_0) = \\ &= v + f'(\varphi)h - \lambda(\langle h, f_0'\rangle f_0 + \langle h, \psi_0'\rangle \psi_0) =: F\end{aligned}$$

Thus,

$$h = (-c\partial_\xi^2 - \lambda\partial_\xi + a)^{-1}[v + f'(\varphi)h - \lambda(\langle h, f_0'\rangle f_0 + \langle h, \psi_0'\rangle \psi_0)],$$

where the operator $(-c\partial_\xi^2 - \lambda\partial_\xi + a)^{-1}$ is given by its multiplier as follows

$$\mathcal{F}[(-c\partial_\xi^2 - \lambda\partial_\xi + a)^{-1}g](k) = \frac{1}{4\pi^2 ck^2 + 2\pi i\lambda k + a}\hat{g}(k).$$

Note that $\|h'\|_{L^2} \leq \|\partial_\xi(-c\partial_\xi^2 - \lambda\partial_\xi + a)^{-1}\|_{L^2\to L^2}\|F\|_{L^2}$. But

$$\begin{aligned}\|F\|_{L^2} &= \|v + f'(\varphi)h - \lambda(\langle h, f_0'\rangle f_0 + \langle h, \psi_0'\rangle \psi_0)\|_{L^2} \leq \\ &\leq \|v\|_{L^2} + \|f'(\varphi)\|_{L^\infty}\|h\|_{L^2} + |\lambda|\|h\|_{L^2}(\|f_0'\|_{L^2} + \|\psi_0'\|_{L^2}) \leq C(\lambda)\|v\|_{L^2},\end{aligned}$$

where in the last line, we have used $\|h\|_{L^2} \leq const.\|v\|_{L^2}$. Finally, by Plancherel's

$$\begin{aligned}\|\partial_\xi(-c\partial_\xi^2 - \lambda\partial_\xi + a)^{-1}\|_{L^2\to L^2} &= \sup_{k\in\mathbf{R}^1}\frac{2\pi|k|}{\sqrt{(4\pi^2ck^2+a)^2 + 4\pi^2\lambda^2k^2}} \leq \sup_{k\in\mathbf{R}^1}\frac{2\pi|k|}{4\pi^2ck^2+a} \\ &\leq \frac{2\pi|k|}{2\sqrt{4\pi^2ck^2a}} = \frac{1}{2\sqrt{ac}}.\end{aligned}$$

Thus, (11) holds. Finally, we set on verifying (10). For $f_0 \in L^1$, one only needs to know that $\int |f'(\varphi(\xi))|d\xi < \infty$, which is assumed. In order to establish $\langle g_0, f_0\rangle = \langle \varphi, f_0\rangle \neq 0$, we proceed similar to Lemma 9. We have from the defining equation of φ and the mean value theorem,

$$\begin{aligned}\langle \mathcal{L}\varphi, \varphi\rangle &= \langle -c\varphi'' + a\varphi - f'(\varphi)\varphi, \varphi\rangle = \langle f(\varphi) - f'(\varphi)\varphi, \varphi\rangle = \\ &= \int \varphi^2(\xi)\left(\int_0^1 [f'(z\varphi(\xi)) - f'(\varphi(\xi))]dz\right) d\xi < 0,\end{aligned}$$

if f is a convex function[12]. Thus $\langle \varphi, f_0\rangle \neq 0$, since otherwise we would have had $\langle \mathcal{L}\varphi, \varphi\rangle \geq 0$. Finally, $\mathcal{L}g_0 = \mathcal{L}\varphi = f(\varphi) - f'(\varphi)\varphi \in L^1 \cap H^1$.

Thus, Theorem 1 applies and the instability would be established, if we can show that $\langle \mathcal{L}^{-1}\psi_0', \psi_0'\rangle = \langle \mathcal{L}^{-1}\varphi'', \varphi''\rangle > 0$. It is clear from the defining equation (45) that φ_c may be written in the form $\varphi_c(x) = \varphi_1(x/\sqrt{c})$, where φ_1 satisfies

$$-\varphi_1'' + a\varphi_1 - f(\varphi_1) = 0.$$

Thus, by taking a derivative with respect to the parameter c in (45), we arrive at

$$\mathcal{L}[\partial_c\varphi_c] - \varphi_c'' = 0,$$

or $\mathcal{L}^{-1}[\varphi_c''] = \partial_c\varphi_c$. As a consequence,

$$\left\langle \mathcal{L}^{-1}\varphi'', \varphi''\right\rangle = \langle \partial_c\varphi_c, \varphi_c''\rangle = -\frac{1}{2}\partial_c\|\varphi_c'\|^2 = -\frac{1}{2}\partial_c[c^{-1/2}\|\varphi_1'\|_{L^2}^2] = \frac{1}{4c^{3/2}}\|\varphi_1'\|_{L^2}^2 > 0.$$

Thus, we obtain that such waves must be spectrally unstable. We have proved the following Theorem.

Theorem 3. *Assume that f is a smooth convex function, $f(u) = O(u^2), f'(u) = O(u)$ for small u. In addition, assume that the solution φ_c to (45) is positive and bell-shaped, with $f'(\varphi_c), f(\varphi) \in L^1(\mathbf{R}^1)$. Then, the wave φ_c is spectrally unstable for all speeds $c > 0$.*

5. Traveling waves for the short pulse equation

In this section, we construct a class of travelling waves for the generalized short pulse equation and show their instability. We would like to mention that there were earlier attempts at constructing such objects, but to the best of our knowledge, the only soliton solutions available for (7) are loop solitons, [28], [17], [15]. These are multivalued functions and as such are not suitable for our purposes. Instead, we construct below a family of (single valued) travelling wave solutions, namely peakons. These are functions continuous at zero, they posses left and right derivative at zero, but their derivative is discontinuous at zero. These are still solutions, in appropriate sense, of (7).

5.1. **Construction of travelling peakons for the generalized short pulse equation.** As we have discussed in the introduction, we consider travelling wave solutions of the generalized short pulse model $u(x,t) = \varphi(x + ct), c > 0$, which yields the equation

$$c\varphi'' = \varphi + (\varphi^p)''. \tag{46}$$

We proceed to construct a class of exact solutions of (46). We will need the precise formulas for the solutions of the ODE

$$-\chi''(x) + \chi(x) - \chi^p(x) = 0, \tag{47}$$

[12] and hence f' is an increasing one

namely

$$\chi(x) = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}}\left(\frac{p-1}{2}x\right). \tag{48}$$

As is well-known, this is the unique positive and even solution of (47). We can rewrite the TW equation (46) in the more convenient for our purposes form

$$[\varphi'(c - p\varphi^{p-1})]' = \varphi. \tag{49}$$

Introduce a new variable η,

$$\xi = \xi(\eta) := \eta - \frac{(p+1)\sqrt{c}}{p-1} \tanh\left(\frac{(p-1)\eta}{2\sqrt{c}}\right). \tag{50}$$

It can be easily checked that the function $\xi(\eta)$ is increasing in $(-\infty, -\eta_{p,c}) \cup (\eta_{p,c}, \infty)$ and decreasing in $(-\eta_{p,c}, \eta_{p,c})$, where

$$\eta_{p,c} = \frac{2\sqrt{c}}{p-1} \ln\left(\frac{\sqrt{2p+2} + \sqrt{2p-2}}{2}\right). \tag{51}$$

Here, $\eta_{p,c}$ is obtained as the positive critical point of $\xi(\eta)$, that is the unique positive solution of the equation

$$\frac{2}{p+1} = \operatorname{sech}^2\left(\frac{(p-1)\eta_{p,c}}{2\sqrt{c}}\right). \tag{52}$$

Note that this allows us to define appropriate inverse functions. Namely, since

$$\xi : (-\infty, -\eta_{p,c}) \to (-\infty, \xi(-\eta_{p,c})),$$
$$\xi : (\eta_{p,c}, \infty) \to (\xi(\eta_{p,c}), \infty),$$

with $\xi(-\eta_{p,c}) > 0$, $\xi(\eta_{p,c}) < 0$, we can define the inverse functions

$$\eta_- : (-\infty, \xi(-\eta_{p,c})) \to (-\infty, -\eta_{p,c}),$$
$$\eta_+ : (\xi(\eta_{p,c}), \infty) \to (\eta_{p,c}, \infty).$$

Next, we seek the solution of the equation $\xi(\eta) = 0$. We find that $\xi(\pm z_{p,c}) = 0$, where

$$z_{p,c} = \frac{2\sqrt{c}}{p-1}\tilde{z}_p : \tanh[\tilde{z}_p] = \frac{2}{p+1}\tilde{z}_p.$$

The constant $\tilde{z}_p$ is well-defined, since the equation $\tanh[z] = \frac{2}{p+1}z$ has only one positive solution, $\tilde{z}_p$. Also, $0 < \eta_{p,c} < z_{p,c}$.

For our purposes, it will be important to consider the restrictions of $\eta_\mp(\xi)$ on the intervals $(-\infty, 0)$ and $(0, \infty)$. We record the fact that both $\eta_\pm$ are increasing functions, which map the appropriate intervals as follows

$$\eta_- : (-\infty, 0) \to (-\infty, -z_{p,c}),$$
$$\eta_+ : (0, \infty) \to (z_{p,c}, \infty).$$

The next step is to introduce an even function $\Phi(\eta)$, which is a solution of the equation

$$\Phi(\eta)(c - p\Phi^{p-1}(\eta)) = c^2\Phi''(\eta). \tag{53}$$

Based on (48), we can write a solution of (53)

$$\Phi(\eta) = \left(\frac{c(p+1)}{2p}\right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}}\left(\frac{(p-1)\eta}{2\sqrt{c}}\right). \tag{54}$$

We take

$$\varphi(\xi) = \begin{cases} \Phi(\eta_-(\xi)) & \xi \le 0 \\ \Phi(\eta_+(\xi)) & \xi > 0. \end{cases}$$

More precisely

$$\varphi(\xi) = \left(\frac{c(p+1)}{2p}\right)^{\frac{1}{p-1}} \begin{cases} \operatorname{sech}^{\frac{2}{p-1}}\left(\frac{(p-1)\eta_-(\xi)}{2\sqrt{c}}\right) & \xi \le 0 \\ \operatorname{sech}^{\frac{2}{p-1}}\left(\frac{(p-1)\eta_+(\xi)}{2\sqrt{c}}\right) & \xi > 0 \end{cases} \tag{55}$$

Since sech is an even function and since $\eta_-(0) = -z_{p,c}$, $\eta_+(0) = z_{p,c}$, the function φ is continuous at zero and it is C^∞ smooth outside zero. That being said, let us proceed to compute the derivative of φ. We have for $\xi \neq 0$,

$$\frac{d\varphi}{d\xi} = \frac{\Phi'(\eta)}{\frac{d\xi}{d\eta}} = \frac{\Phi'(\eta)}{1 - \frac{p+1}{2}\operatorname{sech}^2\left(\frac{(p-1)\eta}{2\sqrt{c}}\right)} = \frac{\Phi'(\eta)}{1 - \frac{p}{c}\Phi^{p-1}(\eta)} = \frac{c\Phi'(\eta)}{c - p\varphi^{p-1}(\xi)}. \tag{56}$$

Here $\eta = \eta_-(\xi)$, if $\xi < 0$ and $\eta = \eta_+(\xi)$, if $\xi > 0$. Note that for all values $\xi \neq \xi(z_{p,c})$, we have that $|\eta| = |\eta_\pm(\xi)| > z_{p,c} > \eta_{p,c}$ and hence the denominator is positive, since

$$\begin{aligned} c - p\Phi^{p-1}(\eta) &> c - p\Phi^{p-1}(z_{p,c}) = c\left(1 - \frac{p+1}{2}\operatorname{sech}^2\left(\frac{(p-1)z_{p,c}}{2\sqrt{c}}\right)\right) > \\ &> c\left(1 - \frac{p+1}{2}\operatorname{sech}^2\left(\frac{(p-1)\eta_{p,c}}{2\sqrt{c}}\right)\right) = 0, \end{aligned}$$

where in the last inequality, we have used that $z_{p,c} > \eta_{p,c}$, $\operatorname{sech}^2(x)$ is decreasing in $(0, \infty)$ and $\eta_{p,c}$ is a solution to (52).

Thus $\varphi'(\xi)(c - p\varphi^{p-1}(\xi)) = \Phi(\eta), \xi \neq \xi(z_{p,c})$, again with the convention that $\eta = \eta_-(\xi)$, if $\xi < 0$ and $\eta = \eta_+(\xi)$, if $\xi > 0$. Differentiating again in $\xi : \xi \neq \xi(z_{p,c})$ yields

$$(\varphi'(c - p\varphi^{p-1}))' = c\frac{\Phi''(\eta)}{\frac{d\xi}{d\eta}} = \frac{c^2\Phi''(\eta)}{c - p\Phi^{p-1}(\eta)}.$$

But according to (53), this last expression equals $\Phi(\eta) = \varphi(\xi)$. Thus, we have shown that φ satisfies (49) for $\xi \neq \xi(z_{p,c})$.

We collect our findings in the following existence result.

Theorem 4. *The equation* (49) *has a peakon solution. More precisely, the function* φ *given by* (55) *is a classical solution for each* $\xi \neq 0$. *In addition,* φ *is* C^∞ *everywhere, except at* $\xi = 0$, *where it is continuous and possesses left and right derivative at zero, but they are different.*

Remark: Note that even at the point $\xi = 0$, which is a point of discontinuity for the function $g(\xi) = \varphi'(c - p\varphi^{p-1}(\xi)$, we have that the right and the left derivatives at $\xi = 0$ exists and they are equal to $\varphi(0)$.

We now consider the linearization around these travelling waves.

5.2. **The linearized problem.** We use the ansatz $u(t,x) = \varphi(x+ct) + v(t, x+ct)$ in the generalized short pulse problem (6). In fact, we rewrite (6) in the form $(u_t - (u^p)_x)_x = u$. Ignoring everything in the form $O(v^2)$, we have

$$(c\varphi' + v_t + cv_x - (\varphi^p + p\varphi^{p-1}v)_x)_x = \varphi + v. \tag{57}$$

By the construction of the solution[13] $[(c - p\varphi^{p-1})\varphi']' = \varphi$ on $(-\infty, 0) \cup (0, \infty)$. Thus, we need to consider the linearized problem

$$(v_t + [(c - p\varphi^{p-1}(\xi))v]_\xi)_\xi = v, \xi \in (-\infty, 0) \cup (0, \infty), \tag{58}$$

where we have adopted the notation $x + ct \to \xi$ for the spatial variable. Next, since we are studying the spectral problem, take $v(t,\xi) = e^{\lambda t} w(\xi)$. We get

$$(\lambda w + [(c - p\varphi^{p-1}(\xi))w]_\xi)_\xi = w, \xi \in (-\infty, 0) \cup (0, \infty). \tag{59}$$

We now introduce a new function z, with $w = z_\xi$. Plugging this in (59) and considering the two intervals $(-\infty, 0)$ and $(0, \infty)$ separately, we obtain

$$\lambda z_\xi + [(c - p\varphi^{p-1}(\xi))z_\xi]_\xi = z + C_1 : \xi \in (-\infty, 0)$$
$$\lambda z_\xi + [(c - p\varphi^{p-1}(\xi))z_\xi]_\xi = z + C_2 : \xi \in (0, \infty).$$

for some constants C_1, C_2. The requirement that z, z_ξ vanishes at $\pm\infty$, imposes on us that $C_1 = C_2 = 0$ and hence the following spectral problem for z,

$$\lambda z_\xi + [(c - p\varphi^{p-1}(\xi))z_\xi]_\xi = z, \xi \in (-\infty, 0) \cup (0, \infty) \tag{60}$$

We change the independent variable as follows. Let Z be the new function, which in terms of the old variable is in the form

$$z(\xi) = \begin{cases} Z(\eta_-(\xi)) & \xi < 0 \\ Z(\eta_+(\xi)) & \xi > 0 \end{cases}$$

Note that the function Z is only defined in $(-\infty, -z_{p,c}) \cup (z_{p,c}, +\infty)$.

Similar to (56), we compute[14]

$$\begin{aligned} z_\xi &= \frac{Z'(\eta)}{\frac{d\xi}{d\eta}} = \frac{Z'(\eta)}{1 - \frac{p+1}{2}\operatorname{sech}^2\left(\frac{(p-1)\eta}{2\sqrt{c}}\right)} = \frac{cZ'(\eta)}{c - p\varphi^{p-1}(\xi)} \\ ((c - p\varphi^{p-1}(\xi))z_\xi)_\xi &= \frac{cZ''(\eta)}{1 - \frac{p+1}{2}\operatorname{sech}^2\left(\frac{(p-1)\eta}{2\sqrt{c}}\right)} = \frac{c^2 Z''(\eta)}{c - p\varphi^{p-1}(\xi)}. \end{aligned}$$

Plugging this in (60) yields the relation

$$\frac{\lambda Z' + c^2 Z''}{c - p\varphi^{p-1}(\xi)} = Z.$$

Taking into account that $\varphi^{p-1}(\xi) = \frac{c(p+1)}{2p}\operatorname{sech}^2\left(\frac{(p-1)\eta}{2\sqrt{c}}\right)$, we arrive at the eigenvalue problem

$$-c^2 Z''(\eta) + cZ(\eta) - p\left[\frac{c(p+1)}{2p}\operatorname{sech}^2\left(\frac{(p-1)\eta}{2\sqrt{c}}\right)\right] Z(\eta) = \lambda Z'(\eta), \tag{61}$$

[13] except at $\xi = 0$

[14] Again $\eta = \eta_\pm(\xi)$

where the variable η only varies in $(-\infty, -z_{p,c}) \cup (z_{p,c}, +\infty)$. That is, the instability of the spectral problem (59) will follow from the solvability of (61) in $(-\infty, -z_{p,c}) \cup (z_{p,c}, +\infty)$. We see that the stability of the travelling peakons for the short pulse equation is reduced to (a non-standard version of) (7). Denote

$$\mathcal{L} = -c^2 \partial_\eta^2 + c - p\left[\frac{c(p+1)}{2p}\operatorname{sech}^2\left(\frac{(p-1)\eta}{2\sqrt{c}}\right)\right]. \tag{62}$$

Lemma 9. *The equation* (61) *has solutions* $Z \in H^2(z_{p,c}, \infty)$ *for any* $\lambda > 0$.

Proof. As we have mentioned earlier, the function $\Phi(\eta) = \left(\frac{c(p+1)}{2p}\operatorname{sech}^2\left(\frac{(p-1)\eta}{2\sqrt{c}}\right)\right)^{\frac{1}{p-1}}$ is the unique even, positive solution to the equation

$$-c^2\Phi'' + c\Phi - p\Phi^p = 0.$$

Denote for conciseness $V(x) = \frac{c(p+1)}{2}\operatorname{sech}^2\left(\frac{(p-1)x}{2\sqrt{c}}\right)$. The linearized operator $\mathcal{L}$ in (62) turns out to be the standard operator L_- in the Schrödinger theory. Recall that as an operator with domain $H^2(\mathbf{R}^1)$, $\mathcal{L} = L_-$ is a non-negative operator, given by the quadratic form

$$q(u,v) = c^2 \langle u', v'\rangle + c\langle u, v\rangle - \langle Vu, v\rangle$$

for $u, v \in D(q) = H^1(\mathbf{R}^1)$. In particular, we have that $q(u,u) \geq 0$ for all $u \in H^1(\mathbf{R}^1)$.

Introduce a new function W in the form $Z(x) = W(x)e^{\mu x}, x > z_{p,c}$. Here $\mu < 0$ will be selected shortly, whence $Z(x)$ will be exponentially localized function at $+\infty$, if W is.

Plugging in $Z(x) = W(x)e^{\mu x}$ in (61) for $x > z_{p,c}$ leads to the following eigenvalue problem

$$-c^2(\mu^2 W + 2\mu W' + W'') + cW - V(x)W = \lambda(\mu W + W'). \tag{63}$$

Clearly, the choice $\mu := -\frac{\lambda}{2c^2} < 0$ will allow us to get rid of W' and leads us to the eigenvalue problem

$$-c^2 W'' + cW - V(x)W = -\frac{\lambda^2}{4c^2}W, \quad x \in (z_{p,c}, \infty). \tag{64}$$

We will construct such a solution W in a straightforward fashion. Let $\sigma := c^{-1}\sqrt{c + \frac{\lambda^2}{4c^2}}$. Then, define Y

$$[L_- + \frac{\lambda^2}{4c^2}]Y = V(x)e^{-\sigma|x|}. \tag{65}$$

This is possible, since $L_- + \frac{\lambda^2}{4c^2} \geq \frac{\lambda^2}{4c^2} > 0$. In addition, note that since[15] $e^{-\sigma|\cdot|} \in H^1(\mathbf{R}^1)$ and the operator $(L_- + \frac{\lambda^2}{4c^2})^{-1} : H^1(\mathbf{R}^1) \to H^3(\mathbf{R}^1)$, we may conclude $Y \in H^3(\mathbf{R}^1)$. In addition, since V is an even potential, the operator $(L_- + \frac{\lambda^2}{4c^2})^{-1}$ acts invariantly on the even subspace of H^1, whence Y is an even function as well.

Writing down explicitly the equation (65), we have

$$-c^2 Y'' + (c + \frac{\lambda^2}{4c^2})Y = V(x)[Y(x) + e^{-\sigma|x|}].$$

[15]In fact $\widehat{e^{-\sigma|\cdot|}} = \frac{2\sigma}{4\pi^2\xi^2 + \sigma^2}$, whence $e^{-\sigma|\cdot|} \in H^{3/2-}$.

At this point, we just take $W(x) := Y(x) + e^{-\sigma|x|}$. Clearly, W is an even function, $W \in H^1(\mathbf{R}^1)$, but note that $W'(x)$ has a jump discontinuity at zero (we show below that this is actually necessary). Also, W is sufficiently smooth in $(-\infty, 0) \cup (0, \infty)$. In addition, for $x \in (-\infty, 0) \cup (0, \infty)$,

$$-c^2 W'' + (c + \frac{\lambda^2}{4c^2})W = -c^2 Y'' + (c + \frac{\lambda^2}{4c^2})Y = V(x)[Y(x) + e^{-\sigma|x|}] = V(x)W(x),$$

since σ was selected so that $(-c^2\partial_x^2 + (c + \frac{\lambda^2}{4c^2}))e^{-\sigma|x|} = 0$ for $x \in (-\infty, 0) \cup (0, \infty)$. Thus, (64) is satisfied in particular $(0, \infty)$, so in $(z_{p,c}, \infty)$ as well. □

Thus, assigning $Z(x) = 0, x < -z_{p,c}$ and Z as in Lemma 9 for $x > z_{p,c}$, we have proved the following result.

Theorem 5. *The peakon solutions φ of the generalized short pulse equation,* (55) *are spectrally unstable for all values of $c > 0$ and $p > 1$.*

Remark: It is worth noting that the (64) has such solutions only because we require this to be in an interval in the form $(z_{p,c}, \infty)$ instead of $\mathbf{R}^1$. In fact, (64) does not have solutions over $(-\infty, 0) \cup (0, \infty)$, for which W' is continuous at zero. Indeed, assuming that it does, it suffices to test such an identity against $W_\epsilon(x) = W(x)(1 - \varphi(x/\epsilon))$ for some $\epsilon > 0$ and a smooth even cutoff function $\varphi : \varphi(x) = 1, |x| < 1$. As $\epsilon \to 0+$, we would obtain[16] $0 \leq q(W, W) = -\frac{\lambda^2}{4c^2}\|W\|^2$, which is a contradiction.

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[16] It is at this limit where we need the continuity of W' at zero, in order to conclude $\langle W'', W_\epsilon \rangle \to -\|W'\|^2$

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