DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS
MATH 220 - SPRING 2005 - FINAL EXAM SOLUTIONS

- **1. (35 points)** Solve the initial-value problem and draw the graph of the solution

$$ty' + 2y = 4t^2 \quad y(1) = 2.$$

Solution:
Since it is first order linear differential equation, we will use integrating factor method. Dividing the equation by $t \neq 0$, we get $y' + \frac{2}{t}y = 4t$. Thus the integrating factor $\mu(t) = e^{\int \frac{2}{t}dt} = t^2$. Multiplying each side of the equation by $\mu(t)$, we get

$$(t^2 y)' = 4t^3$$

If we integrate both sides, we get

$$t^2 y = t^4 + C$$

which implies $y = t^2 + \dfrac{C}{t^2}$. Using initial values, we solve $C = 1$. So the solution to the IVP is

$$y(t) = t^2 + \frac{1}{t^2}$$

- **2. (35 points)** The population of mosquitoes in a certain area increases at a rate proportional to the current population and, in the absence of other factors, the population triples each week. There are 300 000 mosquitoes in the areal initially, and predators (birds, etc.) eat 50 000 mosquitoes a day. Determine the population of mosquitoes in the area at any time.

Solution:

When we model the problem as an initial value problem, we get

$$\left| \begin{array}{l} \frac{dP}{dt} = rP - 50000 \\ P(0) = 300000 \end{array} \right.$$

Using integrating factor method, we get

$$P(t) = P(0)e^{rt} - \frac{50000}{r}(e^{rt} - 1)$$

When there are no predators, the population triples each week, i.e. $P(7) = 3P(0)$. We can replace 50000 by 0. So we get $P(7) = P(0)e^{r7} = 3P(0)$. This implies $e^{7r} = 3$, i.e. $r = \frac{\ln 3}{7} \simeq 0.157$. So the population at time t

$$P(t) = P(0)e^{0.157t} - \frac{50000}{0.157}(e^{0.157t} - 1)$$

- **3. (35 points)** Find the solution of the initial-value problem

$$\left| \begin{array}{l} 2y'' - 3y' + y = 0 \\ y(0) = 2 \\ y'(0) = 1/2 \end{array} \right.$$

Solution:
The characteristic equation is $2r^2 - 3r + 1 = 0$. If we solve for r, we get $r_1 = \frac{1}{2}$ and $r_2 = 1$. So the general solution is

$$y(t) = c_1 e^{t/2} + c_2 e^t$$

Using the initial values, we get $c_1 + c_2 = 2$ and $\dfrac{c_1}{2} + c_2 = 1/2$. We solve for this system and get $c_1 = 3$ and $c_2 = -1$. Thus the solution to the IVP is

$$y(t) = 3e^{t/2} - e^t$$

- **4. (35 points)** Find the general solutions of

(i) $y'' + 4y = 0$; and (ii) $y'' + 4y = \cos(2x) + e^x$.

Solution:
(**i**): The characteristic equation is $r^2 + 4 = 0$. This implies $r = \pm 2i$. Thus the general solution is in the form:

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

(**ii**): This ODE is a non homogenous equation. So we have to find the particular solutions. Let's say y_{p_1} solves $y'' + 4y = \cos(2x)$ and y_{p_2} solves $y'' + 4y = e^x$.

$$y_{p_1} = P(x)\cos(2x) + Q(x)\sin(2x)$$

where $P(x)$ and $Q(x)$ are polynomials. $P(x) = A$ or $P(x) = Ax + B$, $Q(x) = C$ or $Q(x) = Cx + D$ or higher degree polynomials. Let's first assume that $P(x) = A$ and $Q(x) = C$. This is not the case we want, since y_{p_1} will be same as the solution to the homogenous equation. So let's try the other case

$$y_{p_1} = ((Ax + B)\cos(2x) + (Cx + D)\sin(2x))$$

We can drop the term B and D since these terms are represented in the homogenous solution. So we will assume $y_{p_1} = Ax\cos(2x) + Cx\sin(2x)$. Finding y''_{p_1} and plugging into $y''_{p_1} + 4y_{p_1} = \cos(2x)$, we get $A = 0$ and $B = \frac{1}{4}$. So we can conclude that $y_{p_1} = \frac{1}{4}x\sin(2x)$.
Similarly we can assume $y_{p_2} = Ae^x$ and try to find A. Since y_{p_2} satisfies $y''_{p_2} + 4y_{p_2} = e^x$, we get $A = \frac{1}{5}$. Thus the general solution for **(ii)**

$$y(x) = c_1\cos(2x) + c_2\sin(2x) + \frac{1}{4}x\sin(2x) + \frac{1}{5}e^x$$

- **5. (35 points)** Solve the initial-value problem

$$\left| \begin{array}{l} t^3y' + 4t^2y = e^{-t} \\ y(-1) = 0 \end{array} \right.$$

Solution:
First we divide the equation by $t^3 \neq 0$, we get $y' + \frac{4}{t}y = \frac{e^{-t}}{t^3}$.

Using integrating factor, we get $\mu(t) = e^{\int \frac{4}{t}dt} = t^4$. Multiplying the equation by $\mu(t)$, we get

$$t^4y' + 4t^3y = te^{-t}$$

which implies $(t^4y)' = te^{-t}$. If we integrate both sides, we get

$$t^4y = -te^{-t} - e^{-t} + C$$

So the general solution is

$$y = -\frac{1}{t^3}e^{-t} - \frac{1}{t^4}e^{-t} + \frac{C}{t^4}$$

Using the initial values we solve for $C = 0$. So the solution to the IVP is

$$y = -\frac{1}{t^3}e^{-t} - \frac{1}{t^4}e^{-t}$$

- **6. (35 points)** Solve the initial-value problem

$$\left| \begin{array}{l} x + ye^{-x}y' = 0 \\ y(0) = 1 \end{array} \right.$$

Solution:
We will use "Separable Equations Method". If we separate x and y terms, we get

$$-ydy = xe^x dx$$

By integrating both sides, we get

$$-\frac{y^2}{2} = xe^x - e^x + C$$

Using the initial values, we solve for $C = 1/2$. So we get

$$y^2 = -2xe^x + 2e^x - 1$$

Then $y = \pm\sqrt{-2xe^x + 2e^x - 1}$. Since $y(0) = 1$, the solution to the IVP is $y = \sqrt{-2xe^x + 2e^x - 1}$.

- **7. (35 points)** Find the general solution of the linear system

$$\begin{cases} x' &= 2x - 3y \\ y' &= 4x - 6y. \end{cases}$$

Solution:
We can rewrite this system as

(1) $$\mathbf{X}' = \begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix} \mathbf{X}$$

where $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$.
First we find the eigenvalues which satisfy the characteristic equation $\lambda^2 + 4\lambda = 0$. This implies $\lambda_1 = 0$ and $\lambda_2 = -4$. Then we find the corresponding eigenvectors which are $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ respectively. So the general solution is

$$\mathbf{X(t)} = \mathbf{c_1} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \mathbf{c_2 e^{-4t}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- **8. (30 points)** Find the solution of the initial-value problem

$$\begin{cases} x_1' &= x_1 - 5x_2 \\ x_2' &= x_1 - 3x_2. \end{cases}$$

where

$$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Draw the graph of the solution and describe its behavior for increasing t.
Solution:

(2) $$\mathbf{X}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{X}, \qquad \mathbf{x(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.
The eigenvalues satisfy the equation $\lambda^2 + 2\lambda + 2 = 0$. The eigenvalues are $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$. The corresponding eigenvectors are $v_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$.
Then

$$\mathbf{X^1(t)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \mathbf{e^{-t}}(\cos\mathbf{t} + \mathbf{i}\sin\mathbf{t}) = \mathbf{e^{-t}} \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + \mathbf{ie^{-t}} \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$

So the general solution is Then

$$\mathbf{X(t)} = \mathbf{c_1 e^{-t}} \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + \mathbf{c_2 e^{-t}} \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$

If we plug the initial values into the general solution, we get $\begin{array}{ll} 2c_1 + c_2 & = 0 \\ c_1 & = 0 \end{array}$. which implies $c_1 = 0$ and $c_2 = 0$. So the solution is

$$\mathbf{X(t)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- **9. (35 points)** Let

$$\left| \begin{array}{l} y'' - 4y' + 4y = 0 \\ y(0) = 0 \\ y'(0) = a \end{array} \right.$$

find the only value of the parameter a for which the solution stays bounded as $t \to \infty$.
Solution:
The characteristic equation is $(r-2)^2 = 0$ which implies we have a double root $r = 2$. So the general solution is

$$y(t) = c_1 e^{2t} + c_2 t e^{2t}$$

Using the initial values, we solve for $c_1 = 0$ and $c_2 = a$. So the solution the IVP is

$$y(t) = ate^{2t}$$

If we want the solution to be bounded as $t \to \infty$, then $a = 0$. Otherwise it is unbounded.

- **10. (35 points)** Solve the system of equations

$$\left\{ \begin{array}{lll} x_1' & = & x_1 - 4x_2 \\ x_2' & = & 4x_1 - 7x_2 \end{array} \right.$$

Solution:

$$\mathbf{X}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{X} \tag{3}$$

where $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.
The eigenvalues satisfy the equation $\lambda^2 + 6\lambda + 9 = 0$. We have a double eigenvalue at $\lambda = -3$. The corresponding eigenvector is $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

One solution is $\mathbf{X^1(t)} = \mathbf{e^{-3t}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For a second linearly independent solution, we search for a generalized eigenvector. Its components satisfy

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

that is $4\xi_1 - 4\xi_2 = 1$. Let $\xi_2 = k$ for some arbitrary constant. Then $\xi_1 = k + 1/4$. So the second solution

$$\mathbf{X^2(t)} = \mathbf{e^{-3t}t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbf{e^{-3t}} \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} + \mathbf{e^{-3t}k} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Dropping the last term, the general solution is

$$\mathbf{X(t)} = \mathbf{c_1e^{-3t}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbf{c_2} \left(\mathbf{e^{-3t}t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbf{e^{-3t}} \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} \right)$$